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Gauge Fixing of Modified Cubic Open Superstring Field Theory

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Abstract

The gauge-fixing problem in the modified cubic open superstring field theory is discussed in detail for both the Ramond and Neveu-Schwarz (NS) sectors in the Batalin-Vilkovisky (BV) framework. We prove for the first time that the same form of action as the classical gauge-invariant one with the ghost-number constraint on the string field relaxed, gives the master action satisfying the BV master equation. This is achieved by identifying independent component fields by analyzing the kernel structure of the inverse picture-changing operator. The explicit gauge-fixing conditions for the component fields are discussed. In a kind of $b_0 = 0$ gauge, we explicitly obtain an NS propagator that has poles at zeros of the Virasoro operator L_0.

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§1. Introduction

The covariant open superstring field theory (super-SFT) was first constructed by Witten[1)] in the form of Chern-Simons three-form action similarly to his bosonic string field theory.[2)] He wrote the action using string fields in the natural picture, that is, the Neveu-Schwarz (NS) string of picture number -1 and the Ramond (R) string of picture number $-1/2$. The action had a picture-changing operator $X(z)$ located at the midpoint $z = i$ in the cubic NS interaction term. Later, however, it was pointed out by Wendt[3)] that the theory suffers from a severe divergence problem caused by colliding picture-changing operators in the NS four-string amplitude. This led to the violation of the associativity, and hence, the gauge invariance of the theory.

To circumvent this problem, two approaches were proposed. One is the so-called modified cubic super-SFT, which has the same form of action as Witten's but is based on the 0-picture NS string field, and was proposed by Preitschopf, Thorn, and Yost[4)] (PTY) and Arefeva, Medvedev, and Zubarev[5)] (AMZ), independently. Another is Berkovits' super-SFT,[6)] which has the WZW-type nonpolynomial action for the NS sector. The latter approach is interesting in the sense that there is no need for a picture-changing operator. However, in this WZW type approach, the construction of the action for the R sector is difficult and has never been performed in a satisfactory manner.[7)]

The modified cubic super-SFT approach, on the other hand, simply gives the R sector as a cubic action and is beset with no problem. Therefore, we study the modified cubic super-SFT in this paper. The above-mentioned authors who proposed the modified cubic super-SFT discussed the perturbation theory in their papers so that they must have discussed the gauge-fixing problem. However, strangely enough, they did not show that their gauge-fixed action is really BRST-invariant. First, they *tacitly* assumed that the gauge-fixed action, or even the Batalin-Vilkovisky (BV) master action for fields and antifields, takes the *same form* as the original gauge-invariant action with the understanding that the field ghost number constraint is relaxed. However, to the best of our knowledge, no one has ever shown that the 'BV action' thus obtained does really satisfy the BV master equation and hence that the gauge-fixed action obtained by fixing the gauge by setting the suitable set of antifields equal to zero is really BRST-invariant.

The invariance of the 'BV action' under the BRST transformation $\delta_{\mathrm{B}}\Phi = Q_{\mathrm{B}}\Phi + \Phi * \Phi$ is trivial by construction but the BV master equation is not because of the presence of inverse picture-changing operators that have nontrivial kernels and thus are non-invertible. The presence of nontrivial kernels of inverse picture changing operators introduces an additional gauge invariance other than the conventional one. This is important for the very consistency of the R sector in Witten's original super-SFT. This is also the same in this modified cubic super-SFT with the same R sector.

The NS sector is more difficult in which a double-step inverse picture-changing operator is present in the modified cubic super-SFT case. The kernel becomes much larger than that in the R case and the construction of the BRST-invariant projection operator becomes much more complicated accordingly. Because of this complication in the projection operator,[*)] no explicit analysis of component fields of the NS field reduced into the projected space has been carried out; hence, no explicit proof of the BV master equation has been given. The validity of the gauge-fixing procedure could be judged only with it.

The purpose of the present paper is to discuss the gauge fixing of the modified cubic super-SFT, particularly to give an explicit proof that the same form of action as the original gauge-invariant super-SFT action with the ghost number constraint relaxed satisfies the BV master equation. This can be achieved by adopting a simple component field form of the reduced string field suggested directly from the form of the inverse picture-changing operator. This reduced form is not invariant under the BRST transformation, but it can be modified so as to keep the form.

This paper is organized as follows. We first recall the gauge-invariant action of the modified cubic open super-SFT in §2. Then, in order to understand the nature of the problem, we review in §3 the gauge-fixing procedure for the R sector, which was beautifully performed by us and Masaki Murata. This procedure was reported in the string field theory workshops held at APCTP in 2009[8)] and, at YITP in 2010,[9)] but the full report has never been published. The kernel problem in the NS sector is much more complicated than that in the R sector, but we give a solution to this problem in §4, as mentioned above, by defining the simple reduced form of the NS string field for which the double-step picture-changing operator $Y\bar{Y}$ gives a nondegenerate cross-diagonal metric. We show there that it implies that the action satisfies the BV master equation. In §5, we discuss the gauge fixing for the NS sector by explicitly examining the component fields in the reduced NS field. We propose a set of gauge-fixing conditions that gives a BRST-invariant consistent gauge-fixed action. However, since it is difficult to find a propagator in a closed form in that gauge, we propose another gauge in §6 using a nonlocal projection operator. It can give a propagator explicitly, but the explicit component expression becomes much less clear. The gauge is of the $b_0 = 0$ type and the propagator has poles at $L_0 = 0$. A note on the the gauge fixing for the total interacting system is given in §7. The final section, §8, is devoted to discussions.

[*)] Many trials have also been carried out to find simpler BRST-invariant projection operators or even other possibilities of the double-step inverse picture-changing operator for which projection operators become simpler.[9)]

We add three appendices. The explicit expressions for the BRST operators and BRST transformations are given in Appendix A. The definitions and properties of the picture-changing operators are summarized in Appendix B. We give the component expressions for the decomposition by nonlocal projection operators for the R sector in Appendix C.

§2. Gauge-invariant action of modified cubic open super-SFT

The modified cubic open super-SFT action based on the 0-picture NS string $\varPhi$ and $(-1/2)$-picture R string $\varPsi$ was proposed by PTY[4] and AMZ,[5] independently, and is given by

$$\begin{aligned}S &= \frac{1}{2}\int Y\bar{Y}\varPhi * Q_{\mathrm{B}}\varPhi + \frac{1}{3}\int Y\bar{Y}\varPhi * \varPhi * \varPhi + \frac{1}{2}\int Y\varPsi * Q_{\mathrm{B}}\varPsi + \int Y\varPhi * \varPsi * \varPsi \\ &= \frac{1}{2}\langle\varPhi|\, Y\bar{Y}Q_{\mathrm{B}}\,|\varPhi\rangle + \frac{1}{3}\langle V_{\text{NS-NS-NS}}|\, Y\bar{Y}\,|\varPhi\rangle\,|\varPhi\rangle\,|\varPhi\rangle \\ &\qquad\qquad - \frac{1}{2}\langle\bar{\varPsi}|\, YQ_{\mathrm{B}}\,|\varPsi\rangle + \langle V_{\text{NS-R-R}}|\, Y\,|\varPhi\rangle\,|\varPsi\rangle\,|\varPsi\rangle\,. \end{aligned} \tag{2·1}$$

In the classical action the string fields $\varPhi$ and $\varPsi$ are constrained to possess ghost number one.*) The symbol $*$ denotes the conventional Witten's string product based on the midpoint interaction. The operators $Y = Y(i)$ and $\bar{Y} = Y(-i)$ are the inverse picture-changing operators inserted at the string midpoint $z = i$ and its mirror point $z = -i$.**) Note that $\langle\bar{\varPsi}|$ is the Dirac conjugate of $|\varPsi\rangle$:

$$\langle\bar{\varPsi}| = \langle\varPsi|\,\psi_0^0, \qquad \langle\varPsi| = (|\varPsi\rangle)^\dagger, \tag{2·2}$$

where ψ_0^μ is the zero mode of the matter operator $\psi^\mu(z)$ in the R sector.

This system is invariant under the gauge transformations

$$\delta\varPhi = Q_{\mathrm{B}}\lambda + \varPhi * \lambda - \lambda * \varPhi + \bar{X}(\varPsi * \chi - \chi * \varPsi), \tag{2·3a}$$

$$\delta\varPsi = Q_{\mathrm{B}}\chi + \varPhi * \chi - \chi * \varPhi + \varPsi * \lambda - \lambda * \varPsi, \tag{2·3b}$$

where λ (χ) is the NS (R) string transformation parameter with ghost number zero and $\bar{X} = X(-i)$ is the picture-changing operator at the mirror point.

Furthermore, the marked difference from the bosonic string case is that the 'measures' $Y\bar{Y} = Y(i)Y(-i)$ in the NS sector and $Y = Y(i)$ in the R sector have nontrivial kernels since

$$c(z)Y(z) = \gamma^2(z)Y(z) = 0, \tag{2·4}$$

*) This corresponds to the requirement that the coefficient fields carry ghost number zero.

**) Thus we adopt the nonchiral operator as the double-step inverse picture-changing operator following PTY. The conventions of the picture-changing operators are summarized in Appendix B.

as is clear from the expression

$$Y(z) = c(z)\delta'(\gamma(z)). \tag{2·5}$$

Consequently, the kinetic terms of the action (2·1) are also invariant under additional gauge transformations:[10),11)]

$$\delta\Phi = c(i)\lambda_c, \qquad \delta\Phi = \gamma^2(i)\lambda_\gamma, \tag{2·6}$$

$$\delta\Phi = c(-i)\lambda'_c, \qquad \delta\Phi = \gamma^2(-i)\lambda'_\gamma, \tag{2·7}$$

$$\delta\Psi = c(i)\chi_c, \qquad \delta\Psi = \gamma^2(i)\chi_\gamma. \tag{2·8}$$

This is actually the symmetry of the total action (2·1) including the interaction terms because the ghost fields $\gamma(z)$ and $c(z)$ vanish at the interaction point:

$$\langle V_I|\, c(\pm i) = \langle V_I|\, \gamma(\pm i) = 0, \qquad (I = \text{NS-NS-NS or NS-R-R}) \tag{2·9}$$

where $c(\pm i)$ or $\gamma(\pm i)$ can be the ghost coordinate of any of the three strings.

Since the additional gauge invariances are different between the R and NS cases, we discuss the two sectors separately, initially neglecting the Yukawa coupling term $\Phi\Psi\Psi$.

§3. Gauge fixing of Ramond sector

In this section, we discuss the gauge fixing of the gauge symmetries (2·3b) and (2·8) for the R string. This can be performed beautifully. We will provide a prototype solution to the problem.

We first fix the additional gauge invariance (2·8) using the BRST-invariant projection operator $\mathcal{P}_Y$ following Ref. 10):

$$\mathcal{P}_Y \equiv X_0 Y, \qquad X_0 = [Q_{\rm B}, \Theta(\beta_0)] = \delta(\beta_0)F_0 - b_0\delta'(\beta_0), \tag{3·1}$$

where $F_0 = [Q_{\rm B}, \beta_0]$. The choice of X_0 is not unique and has the freedom of changing the gauge for new gauge symmetries. Here, X_0 is the 'mode version' of the picture-changing operator satisfying the properties (B·9) in Appendix B.

The R string field Ψ can be generally expanded in the zero modes of the ghost c_0 and the superghost γ_0 as

$$|\Psi\rangle = \sum_{n=0}^{\infty} (\gamma_0)^n\, |0\rangle_\beta \otimes \Big(|{\downarrow}\rangle \otimes |\phi_n\rangle + c_0\, |{\downarrow}\rangle \otimes |\psi_n\rangle \Big) \tag{3·2}$$

$$\equiv \sum_{n=0}^{\infty} (\gamma_0)^n \Big(|\phi_n\rangle\!\rangle + c_0 |\psi_n\rangle\!\rangle \Big), \tag{3·3}$$

where $|\downarrow\rangle$ and $|0\rangle_\beta$ are, respectively, the vacua of the reparametrization ghost zero mode b_0 and the superghost zero mode β_0 defined by

$$b_0\,|\downarrow\rangle = 0, \qquad \langle\downarrow|\,c_0\,|\downarrow\rangle = 1, \tag{3·4}$$

$$\beta_0\,|0\rangle_\beta = 0, \qquad {}_\beta\langle 0|\,\delta(\gamma_0)\,|0\rangle_\beta = 1. \tag{3·5}$$

The states $|\phi_n\rangle$ and $|\psi_n\rangle$ are the states in the sector other than the (super)ghost zero modes. For brevity, we write $|0\rangle_\beta\otimes|\downarrow\rangle\otimes|\varphi\rangle$ as $|\varphi\rangle\rangle$, which satisfies

$$\langle\langle\varphi_1|c_0\delta(\gamma_0)|\varphi_2\rangle\rangle = \langle\varphi_1|\varphi_2\rangle\,. \tag{3·6}$$

The projected field has been shown[10] to take the same form as the constrained string field proposed in Ref. 12):

$$|\hat{\Psi}\rangle \equiv \mathcal{P}_Y\Psi = |\phi\rangle\rangle - (\gamma_0 + c_0 F)\,|\psi\rangle\rangle, \tag{3·7}$$

where $F = F_0 + 2b_0\gamma_0$ is the Ramond-Dirac operator (A·2c) with the ghost zero modes b_0, γ_0 removed from F_0. Henceforth, the hatted field denotes the field whose kernel degrees of freedom are projected out by the BRST-invariant projection operator $\mathcal{P}_Y$. The Ramond kinetic term can be rewritten with the projected field as

$$\begin{aligned} S[\hat{\Psi}] &= -\frac{1}{2}\langle\bar{\Psi}|\mathcal{P}_Y^T Y Q_{\rm B}\mathcal{P}_Y\,|\Psi\rangle = -\frac{1}{2}\,\langle\Psi|\,\psi_0^0 Y X_0 Y Q_{\rm B} X_0 Y\,|\Psi\rangle \\ &= -\frac{1}{2}\,\langle\Psi|\,Y X_0^\dagger\psi_0^0 Y Q_{\rm B} X_0 Y\,|\Psi\rangle = -\frac{1}{2}\,\langle X_0 Y\Psi|\,\psi_0^0 Y Q_{\rm B} X_0 Y\,|\Psi\rangle \\ &= -\frac{1}{2}\langle\hat{\Psi}|\,\psi_0^0 Y Q_{\rm B}\,|\hat{\Psi}\rangle = -\frac{1}{2}\langle\bar{\hat{\Psi}}|\,Y Q_{\rm B}\,|\hat{\Psi}\rangle\,. \end{aligned} \tag{3·8}$$

Here we have used $\{Y, \psi_0^0\} = 0$ and Eq. (B·14).

Next, for the conventional $Q_{\rm B}$ gauge invariance, we impose the gauge-fixing condition

$$|\psi\rangle = 0. \tag{3·9}$$

Then the projected field $\hat{\Psi}$ satisfies the following two conditions simultaneously:

$$\beta_0\,|\hat{\Psi}\rangle = 0 \quad \text{and} \quad b_0\,|\hat{\Psi}\rangle = 0. \tag{3·10}$$

Let us call this the Ramond-Siegel gauge. We introduce the projection operator into the Ramond-Siegel gauge subspace:

$$\mathcal{P}_G \equiv b_0 c_0 \delta(\beta_0)\delta(\gamma_0), \qquad |\hat{\Psi}_{||}\rangle \equiv \mathcal{P}_G\,|\hat{\Psi}\rangle = |\phi\rangle\rangle. \tag{3·11}$$

3.1. *Iterative gauge-fixing procedure*

For the free action, the gauge fixing can be carried out iteratively just as in the bosonic case.[13)−16)] Let us start from the gauge-invariant free action:

$$S_0 = -\frac{1}{2}\langle\bar{\hat{\Psi}}_{(1)}|Y_0 Q_{\rm B}|\hat{\Psi}_{(1)}\rangle = -\frac{1}{2}\langle\bar{\Psi}_{(1)}|\,\mathcal{P}_Y^T Y_0 Q_{\rm B}\mathcal{P}_Y\,|\Psi_{(1)}\rangle\,, \tag{3·12}$$

where $|\Psi_{(1)}\rangle$ is the classical R string field with ghost number one. Note that the inverse picture-changing operator Y can be rewritten as $Y_0 = c_0\delta'(\gamma_0)$ between the projection operators using Eqs. (B·9):

$$\mathcal{P}_Y^\dagger Y\mathcal{P}_Y = \mathcal{P}_Y^\dagger Y_0\mathcal{P}_Y. \tag{3·13}$$

The above action (3·12) is invariant under the gauge transformation with the gauge parameter of ghost number zero:

$$\delta\,|\hat{\Psi}_{(1)}\rangle = Q_{\rm B}\,|\hat{\Lambda}_{(0)}\rangle\,. \tag{3·14}$$

We can rewrite the gauge transformation (3·14) as the BRST transformation by introducing the Fadeev-Popov (FP) ghost field $|\hat{\Psi}_{(0)}\rangle$ and replacing the parameter $|\hat{\Lambda}_{(0)}\rangle$ with this field as

$$\delta_{\rm B}\,|\hat{\Psi}_{(1)}\rangle = Q_{\rm B}\,|\hat{\Psi}_{(0)}\rangle\,. \tag{3·15}$$

It is convenient to introduce the FP antighost field $|\hat{\Psi}_{(2)}\rangle$ and the Nakanishi-Lautrup (NL) field $|\hat{B}_{(2)}\rangle$ with the BRST transformations

$$\delta_{\rm B}\,|\hat{\Psi}_{(2)}\rangle = |\hat{B}_{(2)}\rangle\,, \qquad \delta_{\rm B}\,|\hat{B}_{(2)}\rangle = 0. \tag{3·16}$$

Taking the Ramond-Siegel gauge condition

$$(1-\mathcal{P}_G)\,|\hat{\Psi}_{(1)}\rangle = 0, \tag{3·17}$$

we add the gauge-fixing and the FP ghost terms to the gauge-invariant action (3·12):

$$\begin{aligned} S_1 &= S_0 - \delta_{\rm B}\left(\langle\bar{\hat{\Psi}}_{(2)}|Y_0(1-\mathcal{P}_G)|\hat{\Psi}_{(1)}\rangle\right) \\ &= -\frac{1}{2}\langle\bar{\hat{\Psi}}_{(1)}|Y_0 Q_{\rm B}|\hat{\Psi}_{(1)}\rangle - \langle\bar{\hat{B}}_{(2)}|Y_0(1-\mathcal{P}_G)|\hat{\Psi}_{(1)}\rangle - \langle\bar{\hat{\Psi}}_{(2)}|\mathcal{P}_G^T Y_0 Q_{\rm B}|\hat{\Psi}_{(0)}\rangle. \end{aligned} \tag{3·18}$$

Here, we have used the equality

$$Y_0(1-\mathcal{P}_G) = \mathcal{P}_G^T Y_0, \tag{3·19}$$

which can be easily confirmed by the following direct calculations:

$$\begin{aligned} Y_0(1-\mathcal{P}_G) &= Y_0 - c_0[\delta(\gamma_0),\beta_0]b_0c_0\delta(\beta_0)\delta(\gamma_0) \\ &= Y_0 + c_0\beta_0\delta(\gamma_0) = c_0\delta(\gamma_0)\beta_0, \end{aligned} \tag{3·20}$$

$$\mathcal{P}_G^T Y_0 = c_0b_0\delta(\gamma_0)\delta(\beta_0)c_0[\delta(\gamma_0),\beta_0] = c_0\delta(\gamma_0)\beta_0. \tag{3·21}$$

This action S_1 is still invariant under the gauge transformation

$$\delta\,|\hat{\Psi}_{(0)}\rangle = Q_{\rm B}\,|\hat{\Lambda}_{(-1)}\rangle\,. \tag{3·22}$$

Repeating the same procedure successively, we can obtain the totally gauge-fixed action as follows. First, we introduce a series of FP ghost fields $|\hat{\Psi}_{(-g)}\rangle$, FP antighost fields $|\hat{\Psi}_{(2+g)}\rangle$, and NL fields $|\hat{B}_{(2+g)}\rangle$ for $g\geq 0$ with the BRST transformations

$$\begin{aligned}
&\delta_{\rm B}\,|\hat{\Psi}_{(1-g)}\rangle = Q_{\rm B}\,|\hat{\Psi}_{(-g)}\rangle\,, \qquad && \delta_{\rm B}\,|\hat{\Psi}_{(2+g)}\rangle = |\hat{B}_{(2+g)}\rangle\,,\\
&\delta_{\rm B}\,|\hat{B}_{(2+g)}\rangle = 0 && \quad \text{for} \quad g\geq 0.
\end{aligned} \tag{3·23}$$

Imposing the Ramond-Siegel gauge condition $(1-\mathcal{P}_G)\,|\hat{\Psi}_{(1-g)}\rangle = 0$ for the FP ghosts and the original field, we obtain

$$\begin{aligned}
S &= -\frac{1}{2}\langle\bar{\hat{\Psi}}_{(1)}\big|Y_0 Q_{\rm B}\big|\hat{\Psi}_{(1)}\rangle - \sum_{g=0}^{\infty}\delta_{\rm B}\left(\langle\bar{\hat{\Psi}}_{(2+g)}\big|Y_0(1-\mathcal{P}_G)\big|\hat{\Psi}_{(1-g)}\rangle\right),\\
&= -\frac{1}{2}\langle\bar{\hat{\Psi}}_{(1)}\big|Y_0 Q_{\rm B}\big|\hat{\Psi}_{(1)}\rangle\\
&\quad -\sum_{g=0}^{\infty}\left(\langle\bar{\hat{B}}_{(2+g)}\big|Y_0(1-\mathcal{P}_G)\big|\hat{\Psi}_{(1-g)}\rangle + \langle\bar{\hat{\Psi}}_{(2+g)}\big|\mathcal{P}_G^T Y_0 Q_{\rm B}\big|\hat{\Psi}_{(-g)}\rangle\right).
\end{aligned} \tag{3·24}$$

The NL fields $|\hat{B}_{(2+g)}\rangle$ serve as gauge-fixing multipliers corresponding to the Ramond-Siegel gauge conditions for the FP ghost fields $|\hat{\Psi}_{(-g)}\rangle$ and the original field $|\hat{\Psi}_{(1)}\rangle$:

$$(1-\mathcal{P}_G)\,|\hat{\Psi}_{(1-g)}\rangle = 0 \quad \text{for} \quad g\geq 0. \tag{3·25}$$

In addition, we can see from the action (3·24) that antighost fields are projected by $\mathcal{P}_G$ and satisfy the Ramond-Siegel gauge condition automatically:

$$(1-\mathcal{P}_G)\mathcal{P}_G\,|\hat{\Psi}_{(2+g)}\rangle = 0 \quad \text{for} \quad g\geq 0. \tag{3·26}$$

Integrating out the NL fields $|\hat{B}_{(2+g)}\rangle$, therefore, we can replace all the string fields $|\hat{\Psi}_{(g)}\rangle$ by the gauge-fixed fields $|\hat{\Psi}_{\|(g)}\rangle = \mathcal{P}_G\,|\hat{\Psi}_{(g)}\rangle$ and obtain

$$\begin{aligned}
S &= -\frac{1}{2}\langle\bar{\hat{\Psi}}_{\|(1)}\big|Y_0 Q_{\rm B}\big|\hat{\Psi}_{\|(1)}\rangle - \sum_{g=0}^{\infty}\langle\bar{\hat{\Psi}}_{\|(2+g)}\big|Y_0 Q_{\rm B}\big|\hat{\Psi}_{\|(-g)}\rangle,\\
&= -\frac{1}{2}\langle\bar{\hat{\Psi}}_{\|}\big|Y_0 Q_{\rm B}\big|\hat{\Psi}_{\|}\rangle,
\end{aligned} \tag{3·27}$$

where

$$|\hat{\Psi}_{\|}\rangle \equiv \sum_{g=-\infty}^{\infty}|\hat{\Psi}_{\|(g)}\rangle\,. \tag{3·28}$$

This action (3·27) has the same form as the original action, but with the ghost number constraint relaxed, and is subject to the gauge condition

$$(1-\mathcal{P}_G)\,|\hat{\Psi}\rangle = 0. \tag{3·29}$$

Because of the equations of motion

$$\mathcal{P}_G\,|\hat{B}_{(2+g)}\rangle = \mathcal{P}_G Q_{\mathrm{B}}\,|\hat{\Psi}_{\parallel(1+g)}\rangle \qquad \text{for} \quad g \geq 0 \tag{3·30}$$

derived by varying the action (3·27) with respect to $|\hat{\Psi}_{\perp(1-g)}\rangle = (1-\mathcal{P}_G)\,|\hat{\Psi}_{(1-g)}\rangle$ the BRST transformations (3·23) can be written in a single BRST transformation law:

$$\delta_{\mathrm{B}}\,|\hat{\Psi}_{\parallel}\rangle = \mathcal{P}_G Q_{\mathrm{B}}\,|\hat{\Psi}_{\parallel}\rangle\,. \tag{3·31}$$

3.2. *Batalin-Vilkovisky (BV) action*

The iterative procedure discussed in the previous subsection is transparent and straightforward but is not applicable in an interacting case. We need to use the BV formalism to fix the nonlinear gauge symmetry (2·3) of the interacting theory.[17),18)] Therefore, here, we first apply the BV formalism in the free R string case and show that it reproduces the results obtained in the previous subsection.

In the BV formalism, we require an extended action $S(\varphi,\varphi^*)$ to satisfy the BV master equation (or Zinn-Justin equation):

$$\sum_i \frac{\partial S}{\partial \varphi^i}\frac{\partial S}{\partial \varphi^*_i} = 0, \tag{3·32}$$

where φ^*_i denotes the antifield conjugate to the field φ^i. If we have such an action, then the gauge fixing can be simply performed by setting the antifields φ^*_i equal to zero. Note that we are adopting the BV formalism in the so-called gauge-fixed basis.[*)] The resultant gauge-fixed action is given by

$$S_{\mathrm{GF}}(\varphi) = S(\varphi,\varphi^*=0), \tag{3·33}$$

which is invariant under the (true) BRST transformation

$$\delta_{\mathrm{B}}\varphi^i = \left.\frac{\partial S}{\partial \varphi^*_i}\right|_{\varphi^*=0}. \tag{3·34}$$

Indeed, it follows from the BV master equation that

$$\delta_{\mathrm{B}} S_{\mathrm{GF}}(\varphi) = \sum_i \left.\frac{\partial S}{\partial \varphi^i}\right|_{\varphi^*=0} \cdot \delta_{\mathrm{B}}\varphi^i \;=\; \left[\sum_i \frac{\partial S}{\partial \varphi^i}\frac{\partial S}{\partial \varphi^*_i}\right]_{\varphi^*=0} = 0. \tag{3·35}$$

[*)] This is because the BV master action in this basis has the same form as the classical gauge-invariant action, as we will see shortly.

Let us apply the BV formalism to the free R string field theory. In the previous subsection, it was shown that the gauge-fixed action has the same form as the classical gauge-invariant action without the ghost number constraint. This suggests that the BV master action also takes the same form:

$$S = -\frac{1}{2}\langle\bar{\hat{\Psi}}|Y_0 Q_{\mathrm{B}}|\hat{\Psi}\rangle, \tag{3·36}$$

which is invariant under the BRST transformation

$$\delta_{\mathrm{B}}|\hat{\Psi}\rangle = Q_{\mathrm{B}}|\hat{\Psi}\rangle\,. \tag{3·37}$$

We now show that this invariance implies that the action (3·36) satisfies the BV master equation (3·35). Using the component form of the projected fields

$$|\hat{\Psi}\rangle = |\phi\rangle\!\rangle - (\gamma_0 + c_0 F)\,|\psi\rangle\!\rangle, \tag{3·38}$$

we can calculate the inner product with the metric Y_0 as

$$\begin{aligned}\langle\bar{\hat{\Psi}}_1|Y_0|\hat{\Psi}_2\rangle &= \big(\langle\!\langle\bar{\phi}_1| - \langle\!\langle\bar{\psi}_1|\gamma_0\big)c_0\delta'(\gamma_0)\big(|\phi_2\rangle\!\rangle - \gamma_0|\psi_2\rangle\!\rangle\big)\\ &= \langle\!\langle\bar{\phi}_1|c_0\delta(\gamma_0)|\psi_2\rangle\!\rangle + \langle\!\langle\bar{\psi}_1|c_0\delta(\gamma_0)|\phi_2\rangle\!\rangle\\ &= \langle\,\bar{\phi}_1|\psi_2\rangle + \langle\,\bar{\psi}_1|\phi_2\rangle\,,\end{aligned} \tag{3·39}$$

where we have used Eq. (3·6). From Eq. (3·39), we can see that the metric Y_0 is just the cross-diagonal nondegenerate metric between the two independent components $|\phi\rangle$ and $|\psi\rangle$ for the projected R field $|\hat{\Psi}\rangle$, just like c_0 in the bosonic string case. Therefore, the general variation δS can be written as

$$\delta S = -\langle\delta\bar{\hat{\Psi}}|Y_0 Q_{\mathrm{B}}|\hat{\Psi}\rangle = -\langle\delta\bar{\hat{\Psi}}|Y_0|\delta_{\mathrm{B}}\hat{\Psi}\rangle = \langle\,\delta\bar{\phi}|\delta_{\mathrm{B}}\psi\rangle + \langle\,\delta\bar{\psi}|\delta_{\mathrm{B}}\phi\rangle\,, \tag{3·40}$$

with

$$\delta_{\mathrm{B}}|\hat{\Psi}\rangle = |\delta_{\mathrm{B}}\phi\rangle\!\rangle - (\gamma_0 + c_0 F)\,|\delta_{\mathrm{B}}\psi\rangle\!\rangle. \tag{3·41}$$

We thus have*)

$$\frac{\partial S}{\partial\psi} = \delta_{\mathrm{B}}\phi, \qquad \frac{\partial S}{\partial\phi} = \delta_{\mathrm{B}}\psi. \tag{3·42}$$

As a result, the BRST invariance of the master action implies that

$$0 = \delta_{\mathrm{B}}S = \frac{\partial S}{\partial\phi}\delta_{\mathrm{B}}\phi + \frac{\partial S}{\partial\psi}\delta_{\mathrm{B}}\psi = 2\frac{\partial S}{\partial\phi}\frac{\partial S}{\partial\psi}, \tag{3·43}$$

*) The fields ϕ and ψ in Eqs. (3·42) actually represent the coefficient fields appearing in the expansion of the string states $|\phi\rangle$ and $|\psi\rangle$, respectively. Since $\langle\,\delta\bar{\phi}| = \langle\delta\phi|\,\psi_0^0$ and ψ_0^0 is fermionic, these coefficient fields ϕ and ψ have a nonvanishing inner product between those with opposite statistics, as it should be for the BV field and antifield. Although $\bar{\phi} = \phi^T C$ for the coefficient fields, we take $C = 1$, which is possible for GSO-projected chiral fields.

which is nothing but the BV master equation, if we identify ϕ and ψ as the field and antifield, respectively. In this identification, the BV gauge fixing defined by setting the antifield equal to zero gives the Ramond-Siegel gauge $|\psi\rangle = 0$. The BV gauge-fixed action (3·33) and the BRST transformation

$$\delta_{\mathrm{B}}\phi = \frac{\partial S}{\partial\psi}\bigg|_{\psi=0} = \tilde{Q}_{\mathrm{B}}\phi \tag{3·44}$$

respectively coincide with the previous results (3·27) and (3·31).

3.3. *Propagator*

Before closing this section, we derive the propagator of the R string for the gauge condition (3·10). Let us consider the gauge-fixed free action (3·27) with a source J:

$$\begin{aligned} S[\hat{\Psi}_{\|}, J] &= -\frac{1}{2}\langle\bar{\hat{\Psi}}_{\|}|Y_0Q_{\mathrm{B}}|\hat{\Psi}_{\|}\rangle + \langle\bar{\hat{\Psi}}_{\|}|J\rangle \\ &= -\frac{1}{2}\langle\bar{\Psi}|\mathcal{P}_Y^T\mathcal{P}_G^TY_0Q_{\mathrm{B}}\mathcal{P}_G\mathcal{P}_Y\,|\Psi\rangle + \langle\bar{\Psi}|\mathcal{P}_Y^T\mathcal{P}_G^T\,|J\rangle\,. \end{aligned} \tag{3·45}$$

The propagator can be found by eliminating $|\hat{\Psi}_{\|}\rangle$ using the following equation of motion:

$$\mathcal{P}_Y^T\mathcal{P}_G^T\left(Y_0Q_{\mathrm{B}}\,|\hat{\Psi}_{\|}\rangle - |J\rangle\right) = 0. \tag{3·46}$$

A solution to this is given by

$$|\hat{\Psi}_{\|}\rangle = \frac{b_0X_0}{L_0}\,|J\rangle = b_0\frac{\delta(\beta_0)}{F_0}\,|J\rangle\,. \tag{3·47}$$

Indeed, using the relation (3·21) and $F_0^2 = L_0$, we can confirm this as follows:

$$\begin{aligned} \mathcal{P}_G^TY_0Q_{\mathrm{B}}\frac{b_0X_0}{L_0}\,|J\rangle &= c_0\delta(\gamma_0)\beta_0Q_{\mathrm{B}}\frac{b_0\delta(\beta_0)}{F_0}\,|J\rangle \\ &= c_0\delta(\gamma_0)(-F_0)b_0\delta(\beta_0)\frac{1}{F_0}\,|J\rangle \\ &= c_0b_0\delta(\gamma_0)\delta(\beta_0)\,|J\rangle = \mathcal{P}_G^T\,|J\rangle\,. \end{aligned} \tag{3·48}$$

Substituting this expression back into the action (3·45), we obtain

$$\begin{aligned} S[\hat{\Psi}_{\|}(J), J] &= -\frac{1}{2}\langle\bar{J}|\,\frac{X_0b_0}{L_0}Y_0Q_{\mathrm{B}}\frac{b_0X_0}{L_0}\,|J\rangle + \langle\bar{J}|\,\frac{X_0b_0}{L_0}\,|J\rangle \\ &= -\frac{1}{2}\langle\bar{J}|\,\frac{b_0X_0}{L_0}\,|J\rangle\,. \end{aligned} \tag{3·49}$$

Thus, the Ramond propagator Π_{R} is given by

$$\Pi_{\mathrm{R}} = \frac{b_0X_0}{L_0} = b_0\frac{\delta(\beta_0)}{F_0}. \tag{3·50}$$

§4. Reduced form of NS field and BV master equation

We concentrate in this section on the NS sector action:

$$S = \frac{1}{2}\int Y\bar{Y}\,\Phi * Q_{\rm B}\Phi + \frac{1}{3}\int Y\bar{Y}\,\Phi * \Phi * \Phi. \tag{4·1}$$

4.1. *Resolving the kernel problem of $Y\bar{Y}$*

Owing to the kernel of the inverse picture-changing operator $Y\bar{Y}$, if we expand the NS fields in powers of the ghost and superghost factors, $c(i),\ c(-i)$ and $\gamma(i),\ \gamma(-i)$, the terms containing $c(i),\ c(-i)$ and $\gamma^n(i),\ \gamma^n(-i)$ $(n \geq 2)$ can all be eliminated. In connection with this, we here define the following combinations of the ghost and superghost variables $c(\pm i)$ and $\gamma(\pm i)$ and note their mode expansions:

$$\begin{aligned} c_\pm(i) &\equiv \frac{1}{2}\begin{pmatrix} c(i)+c(-i) \\ i^{-1}(c(i)-c(-i)) \end{pmatrix} = \begin{pmatrix} c_- \\ c_0 \end{pmatrix} + C_\pm(i) = e^{-T}\begin{pmatrix} c_- \\ c_0 \end{pmatrix} e^{T}, \\ \gamma_\pm(i) &\equiv \frac{1}{2}\begin{pmatrix} \gamma(i)+\gamma(-i) \\ i^{-1}(\gamma(i)-\gamma(-i)) \end{pmatrix} = \gamma_{\pm\frac{1}{2}} + \Gamma_\pm(i) = e^{-T}\begin{pmatrix} \gamma_{+\frac{1}{2}} \\ \gamma_{-\frac{1}{2}} \end{pmatrix} e^{T}, \end{aligned} \tag{4·2}$$

where $c_- = c_1 - c_{-1}$, and

$$T = b_0 C_- + \frac{1}{2}b_- C_+ + \beta_{-\frac{1}{2}}\Gamma_+ + \beta_{\frac{1}{2}}\Gamma_-, \qquad b_- = \frac{1}{2}(b_{-1}-b_1), \tag{4·3}$$

$$\begin{aligned} C_+(i) &= \sum_{k=1}^{\infty}(-1)^k(c_{2k+1} - c_{-(2k+1)}), \quad C_-(i) = \sum_{k=1}^{\infty}(-1)^k(c_{2k} + c_{-2k}), \\ \Gamma_+(i) &= \sum_{k=1}^{\infty}(-1)^k(\gamma_{2k+\frac{1}{2}} + \gamma_{-2k+\frac{1}{2}}), \quad \Gamma_-(i) = \sum_{k=1}^{\infty}(-1)^k(\gamma_{2k-\frac{1}{2}} + \gamma_{-2k-\frac{1}{2}}). \end{aligned} \tag{4·4}$$

Then, in front of the measure $Y\bar{Y}$, the general NS string field Φ can always be rewritten into the following projected form (noting that $\gamma_+^2(i) + \gamma_-^2(i) = \gamma(i)\gamma(-i)$):*)

$$\begin{aligned} \mathcal{P}_0\Phi &\equiv \widetilde{\Phi} \\ &\equiv |\phi_0\rangle + \gamma_+(i)\,|\phi_+\rangle + \gamma_-(i)\,|\phi_-\rangle + \frac{1}{2}(\gamma_+^2(i)+\gamma_-^2(i))\,|\phi_{+-}\rangle \\ &\quad + c_+\Big(|\psi_0\rangle + \gamma_+(i)\,|\psi_+\rangle + \gamma_-(i)\,|\psi_-\rangle + \frac{1}{2}(\gamma_+^2(i)+\gamma_-^2(i))\,|\psi_{+-}\rangle\Big), \end{aligned} \tag{4·5}$$

where $c_+ \equiv c_1 + c_{-1}$ and all the component fields are annihilated by $b_0, b_{\pm1}, \beta_{\pm\frac{1}{2}}$:

$$\big(b_0,\ b_{\pm1},\ \beta_{\pm\frac{1}{2}}\big)\big(|\phi_0\rangle\,,\ |\phi_\pm\rangle\,,\ |\phi_{+-}\rangle\,,\ |\psi_0\rangle\,, |\psi_\pm\rangle\,, |\psi_{+-}\rangle\big) = 0. \tag{4·6}$$

*) These states are therefore the tensor product of the superconformal vacuum $|0\rangle$ in the $(\beta_{\pm\frac{1}{2}}, \gamma_{\pm\frac{1}{2}}; b_{0,\pm1}, c_{0,\pm1})$ sector and the states $|\phi_0\rangle\,, |\psi_0\rangle\,, \cdots$ in the other mode sector. Therefore, if we follow the notation in §3, these states should be written as $|\phi_0\rangle\!\rangle, |\psi_0\rangle\!\rangle, \cdots$, but here we omit the distinction between $|\ \ \rangle$ and $|\ \ \rangle\!\rangle$ for simplicity.

Since $e^T\gamma_\pm(i)\,e^{-T}=\gamma_{\pm\frac{1}{2}}$, the expansion in Eq. (4·5) is in fact the mode expansion in powers of $\gamma_{\pm\frac{1}{2}}$ if transformed by the translation operator e^T:

$$\begin{aligned} e^T\mathcal{P}_0\Phi &= |\phi_0\rangle + \gamma_{+\frac{1}{2}}\,|\phi_+\rangle + \gamma_{-\frac{1}{2}}\,|\phi_-\rangle + \frac{1}{2}(\gamma^2_{+\frac{1}{2}}+\gamma^2_{-\frac{1}{2}})\,|\phi_{+-}\rangle \\ &\quad + c_+\Big(|\psi_0\rangle + \gamma_{+\frac{1}{2}}\,|\psi_+\rangle + \gamma_{-\frac{1}{2}}\,|\psi_-\rangle + \frac{1}{2}(\gamma^2_{+\frac{1}{2}}+\gamma^2_{-\frac{1}{2}})\,|\psi_{+-}\rangle\Big). \end{aligned} \tag{4·7}$$

The explicit expression of the projection operator $e^T\mathcal{P}_0\,e^{-T}$ can be given as

$$\begin{aligned} e^T\mathcal{P}_0\,e^{-T} = b_-c_-\Big[&\delta^2(\beta)\delta^2(\gamma) - \gamma_{\frac{1}{2}}\delta^2(\beta)\delta^2(\gamma)\beta_{-\frac{1}{2}} - \gamma_{-\frac{1}{2}}\delta^2(\beta)\delta^2(\gamma)\beta_{\frac{1}{2}} \\ &+\frac{1}{2}\left(\gamma^2_{\frac{1}{2}}+\gamma^2_{-\frac{1}{2}}\right)\delta^2(\beta)\delta^2(\gamma)\frac{1}{2}\left(\gamma^2_{\frac{1}{2}}+\gamma^2_{-\frac{1}{2}}\right)\Big], \end{aligned} \tag{4·8}$$

$$\delta^2(\beta)=\delta(\beta_{\frac{1}{2}})\delta(\beta_{-\frac{1}{2}}),\quad \delta^2(\gamma)=\delta(\gamma_{\frac{1}{2}})\delta(\gamma_{-\frac{1}{2}}). \tag{4·9}$$

In front of the $Y\bar{Y}$, we can always reduce the field into the projected form (4·5). We adopt this reduction as our convention for the NS field, then the kernel problem of $Y\bar{Y}$ is resolved. The operator $Y\bar{Y}$ in fact gives a nondegenerate and cross-diagonal metric in this projected subspace. Indeed, noting that

$$e^TY\bar{Y}\,e^{-T}\equiv(Y\bar{Y})_0=\frac{1}{4}c_-c_0\left(\delta''(\gamma_{+\frac{1}{2}})\delta(\gamma_{-\frac{1}{2}})+\delta(\gamma_{+\frac{1}{2}})\delta''(\gamma_{-\frac{1}{2}})\right), \tag{4·10}$$

and $\delta''(\gamma)=\big[[\delta(\gamma),\,\beta],\,\beta\big]=\beta^2\delta(\gamma)-2\beta\delta(\gamma)\beta+\delta(\gamma)\beta^2$ for $\gamma=\gamma_{\pm\frac{1}{2}}$ and $\beta=\beta_{\mp\frac{1}{2}}$ satisfying $[\gamma,\,\beta]=1$, we find that the inner product of the two NS fields $\Phi^{(1)}$ and $\Phi^{(2)}$ written in the form (4·5) is given by

$$\begin{aligned} \big\langle\Phi^{(1)}\big|\,Y\bar{Y}\,\big|\Phi^{(2)}\big\rangle &= \big\langle\widetilde{\Phi}^{(1)}\big|Y\bar{Y}\big|\widetilde{\Phi}^{(2)}\big\rangle = \big\langle\tilde{\Phi}^{(1)}\big|\,e^{-T}(Y\bar{Y})_0\,e^T\big|\tilde{\Phi}^{(2)}\big\rangle \\ &= \big\langle\phi_0^{(1)}\big|\psi_{+-}^{(2)}\big\rangle + \big\langle\phi_+^{(1)}\big|\psi_-^{(2)}\big\rangle + \big\langle\phi_-^{(1)}\big|\psi_+^{(2)}\big\rangle + \big\langle\phi_{+-}^{(1)}\big|\psi_0^{(2)}\big\rangle + (\phi\leftrightarrow\psi). \end{aligned} \tag{4·11}$$

This cross-diagonal form of the inner product is very important for the BV master equation below.

4.2. *BV action satisfying the master equation*

We claim that the action satisfying the BV master equation takes the same form as the classical gauge-invariant action with the ghost number constraint relaxed.

The gauge invariance of the original classical action was realized by the derivation property and partial integrability of the BRST operator Q_B on the star product as well as the associativity of the star product. Then the present action with ghost number constraint on Φ relaxed can also be seen as invariant under the following BRST transformation δ_B:

$$\delta_\mathrm{B}\Phi=Q_\mathrm{B}\Phi+\Phi*\Phi. \tag{4·12}$$

The action S is constructed such that it takes a particular form for any variation $\delta\Phi$:

$$\delta S = \int Y\bar{Y}\delta\Phi * \delta_{\rm B}\Phi = \langle\delta\Phi|\, Y\bar{Y}\, |\delta_{\rm B}\Phi\rangle\,. \tag{4·13}$$

Note that the presence of $Y\bar{Y}$ does not injure the partial integrability and derivation property of the BRST operator $Q_{\rm B}$ because $[Q_{\rm B}, Y\bar{Y}] = 0$ nor the associativity of the star product since $Y\bar{Y}$ is placed at the midpoint that is common to all the participating strings. Therefore, the action is invariant under the BRST transformation

$$\delta_{\rm B}S = \int Y\bar{Y}\delta_{\rm B}\Phi * \delta_{\rm B}\Phi = 0. \tag{4·14}$$

However, it is not quite trivial that this BRST invariance of the system guarantees the BV master equation of the action S because of the presence of the $Y\bar{Y}$ factor. It has the kernel so that we projected the NS field Φ into the reduced form $\mathcal{P}_0\Phi = \widetilde{\Phi}$, as given in Eq. (4·5). But the above BRST transformation $\delta_{\rm B}\Phi$ does *not* give a closed transformation in such a space of reduced form string fields $\widetilde{\Phi}$. Fortunately, however, the $Y\bar{Y}$ factor in the action remains present in front even after the BRST transformation, so that it is automatically projected into the reduced form. That is, we can define the BRST transformation in the reduced space as

$$\delta_{\rm B}\widetilde{\Phi} = \mathcal{P}_0(Q_{\rm B}\widetilde{\Phi} + \widetilde{\Phi} * \widetilde{\Phi}). \tag{4·15}$$

Here, we have written all Φ by $\widetilde{\Phi}$ since we regard the reduced field $\widetilde{\Phi}$ as our basic variable, and actually only those components appear in the action because of the presence of the $Y\bar{Y}$ factor.*) Equation (4·13) can be rewritten in the form

$$\delta S = \langle\delta\widetilde{\Phi}|\, Y\bar{Y}\, |\delta_{\rm B}\widetilde{\Phi}\,\rangle = \big\langle\, e^T\delta\widetilde{\Phi}\big|(Y\bar{Y})_0\big|\, e^T\delta_{\rm B}\widetilde{\Phi}\big\rangle. \tag{4·16}$$

However, we know that the metric $Y\bar{Y}$ is cross-diagonal in the reduced NS string field space, so that we have

$$\delta S = \big\langle\delta\phi_0\big|\delta_{\rm B}\psi_{+-}\big\rangle + \big\langle\delta\phi_+\big|\delta_{\rm B}\psi_-\big\rangle + \big\langle\delta\phi_-\big|\delta_{\rm B}\psi_+\big\rangle + \big\langle\delta\phi_{+-}\big|\delta_{\rm B}\psi_0\big\rangle + (\phi\leftrightarrow\psi); \tag{4·17}$$

hence,

$$\delta_{\rm B}\psi_i = \frac{\partial S}{\partial\phi_{I(i)}},\quad \delta_{\rm B}\phi_i = \frac{\partial S}{\partial\psi_{I(i)}},\qquad \text{with}\quad i = \begin{pmatrix} +- \\ \mp \\ 0 \end{pmatrix} \leftrightarrow I(i) = \begin{pmatrix} 0 \\ \pm \\ +- \end{pmatrix}. \tag{4·18}$$

*) This, in fact, holds even in the absence of the $Y\bar{Y}$ factor for the interaction term, since the ghost $c(\pm i)$ and superghost $\gamma(\pm i)$ at the midpoint vanish on Witten's three-string vertex even without the $Y\bar{Y}$ factor, Eq. (2·9).

With these relations, the BRST invariance of the action implies that the action S satisfies the BV master equation; indeed, we have

$$0 = \delta_{\mathrm{B}} S = \sum_{i=0,\pm,+-} \left(\frac{\partial S}{\partial \phi_i}\, \delta_{\mathrm{B}} \phi_i + \frac{\partial S}{\partial \psi_i}\, \delta_{\mathrm{B}} \psi_i \right) = \sum_i \left(\frac{\partial S}{\partial \phi_i} \frac{\partial S}{\partial \psi_{I(i)}} + \frac{\partial S}{\partial \psi_i} \frac{\partial S}{\partial \phi_{I(i)}} \right), \tag{4·19}$$

so that

$$\sum_i \frac{\partial S}{\partial \phi_i} \frac{\partial S}{\partial \psi_{I(i)}} = 0. \tag{4·20}$$

§5. Gauge fixing of NS sector by component fields

Now that we have shown that our action S satisfies the BV master equation, we can discuss how to fix the gauge explicitly in component fields.

The gauge can generally be fixed by setting the antifields equal to zero. However, the choice of the gauge-fixing conditions is of course not unique. Accordingly, the choice of the set of the antifields is also not unique, and there is actually the freedom of doing graded canonical transformation of field and antifield variables. However, we should note that the latter freedom of graded canonical transformation is much wider*) and that *not* all the sets of antifields can be used as gauge-fixing conditions. To find suitable sets of antifields to be set equal to zero as gauge fixing, we have to confirm that they can actually be set equal to zero by the BRST transformation (which contains the original gauge transformation for the original gauge fields).

To understand the situation better, let us recall the case of bosonic SFT as the simplest example of gauge fixing. The bosonic string field is expanded in the FP ghost zero mode c_0 as $\varPhi = \phi + c_0 \psi$ and the action S satisfies the BV master equation

$$\frac{\partial S}{\partial \phi} \frac{\partial S}{\partial \psi} = 0. \tag{5·1}$$

Since this BV master equation is totally symmetric between ϕ and ψ, we cannot determine from this equation alone which variable ϕ or ψ should be taken as the antifield set to be zero. To determine it properly, we need to have a closer look into the (free part of the) BRST transformation $\delta_{\mathrm{B}} \varPhi = Q_{\mathrm{B}} \varPhi = (c_0 L_0 + b_0 M + \tilde{Q}_{\mathrm{B}}) \varPhi$, which takes the forms of the component fields ϕ and ψ:

$$\delta_{\mathrm{B}} \phi = \tilde{Q}_{\mathrm{B}} \phi + M \psi, \tag{5·2}$$
$$\delta_{\mathrm{B}} \psi = -L_0 \phi + \tilde{Q}_{\mathrm{B}} \psi. \tag{5·3}$$

*) Indeed, even exchanging the fields and antifields is contained as a special graded canonical transformation, as is well known.

From this expression, we understand why we can take ψ but not ϕ as the antifield, since ψ can be set equal to zero using the $-L_0\phi$ part in $\delta_{\rm B}\psi$. Note that the Klein-Gordon Virasoro operator L_0 is regarded as invertible in this discussion of general off-shell fields.*) This is the moral of the game that we fully use below. However, the operators $\tilde{Q}_{\rm B}$ and M appearing on the RHS of $\delta_{\rm B}\phi$ are non-invertible; thus, ϕ cannot totally be gauged away. In addition, we know that the physical gauge field is contained in ϕ, which also supports the observation that ϕ cannot be eliminated.

Now we go back to our discussion of the present problem of the NS string.

We first show that the $b_+\widetilde{\Phi}=0$ gauge adopted by PTY is invalid:

$$b_+\widetilde{\Phi}=0 \quad \leftrightarrow \quad \psi_0=\psi_+=\psi_-=\psi_{+-}=0. \tag{5·4}$$

The reason why this gauge is not good is that not only is its perturbation theory very singular, but the gauge itself cannot be taken. Indeed, the ψ_0 component contains, for instance, the original (ghost-number zero) physical gauge field, which clearly cannot be gauged away. As was shown by Urosevic and Zubarev[19] explicitly, $|\psi_0\rangle$ contains

$$|\psi_0\rangle = \frac{1}{2}A_\mu(k)\alpha_{-1}^{\mu}\,e^{ikx}\,|0\rangle\,, \tag{5·5}$$

which is identified as the massless gauge field. Indeed the BRST transformation $\delta_{\rm B}\widetilde{\Phi}=\mathcal{P}_0 Q_{\rm B}\widetilde{\Phi}$ for the component $|\psi_0\rangle$ is given by

$$\delta_{\rm B}\,|\psi_0\rangle = -L_+\,|\phi_0\rangle+\cdots\,, \tag{5·6}$$

but only the part $\frac{1}{2}p_\mu\alpha_{-1}^{\mu}$ in L_+ with $|\phi_0\rangle=\lambda(k)\,e^{ikx}\,|0\rangle$ can contribute to the transformation and takes the form of the conventional gauge transformation

$$\delta_{\rm B}A_\mu(k)=k_\mu\lambda(k). \tag{5·7}$$

Namely, the transverse part of the gauge field A_μ at least cannot be gauged away. This also demonstrates that the impossibility of eliminating $|\psi_0\rangle$ is essentially connected to the non-invertibility of the operator L_+. If it were invertible, the $-L_+\,|\phi_0\rangle$ part in $\delta_{\rm B}\,|\psi_0\rangle$ could have eliminated everything in $|\psi_0\rangle$. Therefore this explains the reason why the impossibility of the $b_+=0$ gauge is related to the singular perturbation theory using the propagator b_+/L_+.

Now, we consider the BRST transformation $\delta_{\rm B}\widetilde{\Phi}$ for all the eight component fields $|\phi_0\rangle\,,|\phi_\pm\rangle\,,|\phi_{+-}\rangle$ and $|\psi_0\rangle\,,|\psi_\pm\rangle\,,|\psi_{+-}\rangle$, the explicit form of which is given in Appendix A.

*) Even the on-shell $L_0=0$ component of ψ can be gauged away by using the 'dipole-ghost' component in ϕ, satisfying $L_0\phi=-\psi\neq 0$ and $L_0^2\phi=0$.

Since it is simpler if we eliminate higher-power terms in $\gamma_{\pm\frac{1}{2}}$, we first try to eliminate $|\psi_{+-}\rangle$. We note that the RHS of $\delta_{\mathrm{B}}|\psi_{+-}\rangle$ contains $(G_{-\frac{1}{2}}|\psi_{+}\rangle + G_{\frac{1}{2}}|\psi_{-}\rangle)/2$ so, if $|\psi_{\pm}\rangle$ contain the component of the form $G_{\pm\frac{1}{2}}|\psi\rangle$ with a common $|\psi\rangle$, then it yields

$$\frac{1}{2}\Big(G_{-\frac{1}{2}}|\psi_{+}\rangle + G_{\frac{1}{2}}|\psi_{-}\rangle\Big) = \frac{1}{2}\big\{G_{-\frac{1}{2}},\, G_{+\frac{1}{2}}\big\}|\psi\rangle = L_0|\psi\rangle\,. \tag{5·8}$$

Since L_0 is invertible as emphasized above, this freedom $L_0|\psi\rangle$ can totally eliminate the $|\psi_{+-}\rangle$ component. In the same way, we can eliminate $|\phi_{+-}\rangle$ using the part

$$\frac{1}{2}\Big(G_{-\frac{1}{2}}|\phi_{+}\rangle + G_{\frac{1}{2}}|\phi_{-}\rangle\Big) \quad \text{or} \quad -2|\psi_0\rangle \tag{5·9}$$

contained in $\delta_{\mathrm{B}}|\phi_{+-}\rangle$. Thus, we have seen that we can take the gauge

$$|\psi_{+-}\rangle = |\phi_{+-}\rangle = 0\,. \tag{5·10}$$

Under these gauge conditions, $|\psi_{+-}\rangle$ and $|\phi_{+-}\rangle$ are now identified as antifields and the components $|\phi_0\rangle$ and $|\psi_0\rangle$ are the fields conjugate to them, respectively, as seen from the metric structure (4·17).

Next, looking at the transformation law $\delta_{\mathrm{B}}|\psi_{\pm}\rangle$, we find the part $\pm\frac{1}{2}|\phi_{\mp}\rangle$, so that we are tempted to choose the gauge setting $|\psi_{\pm}\rangle = 0$. However, if we do so, all the terms on the RHS of $\delta_{\mathrm{B}}|\psi_{+-}\rangle$ vanish. This implies by Eq. (4·17) that the field component $|\phi_0\rangle$ does not appear at all in the kinetic term in the action. Thus the kinetic term is singular so that it is not allowed. This observation is consistent with the statement that we should keep the component $G_{-\frac{1}{2}}|\psi_{+}\rangle + G_{\frac{1}{2}}|\psi_{-}\rangle$ nonzero so that we can use it to eliminate the $|\psi_{+-}\rangle$ component. The lesson here is that the RHS of the BRST transformation of the *antifield*, giving an equation of motion of the gauge-fixed action, should contain *field* components not eliminated by gauge fixing.

To find suitable gauge-fixing conditions for the sector of $(|\phi_{\pm}\rangle, |\psi_{\pm}\rangle)$, let us introduce the following decomposition of the components $|\phi_{\pm}\rangle$ into $|\phi\rangle$ and $|\phi^*\rangle$:

$$|\phi_{\pm}\rangle = G_{\pm\frac{1}{2}}|\phi\rangle + G_{\mp\frac{1}{2}}|\phi^*\rangle\,. \tag{5·11}$$

Conversely, these $|\phi\rangle$ and $|\phi^*\rangle$ can be expressed in terms of $|\phi_{\pm}\rangle$ as follows. Multiplying $G_{\pm\frac{1}{2}}$ to (5·11) and using $\big(G_{\frac{1}{2}}\big)^2 + \big(G_{-\frac{1}{2}}\big)^2 = 2L_{+}$ and $\{G_{\frac{1}{2}}, G_{-\frac{1}{2}}\} = 2L_0$, we obtain

$$\begin{aligned} |G_{\mp}\phi_{\pm}\rangle &= L_0|\phi\rangle + L_{+}|\phi^*\rangle\,, \\ |G_{\pm}\phi_{\pm}\rangle &= L_{+}|\phi\rangle + L_0|\phi^*\rangle\,, \end{aligned} \tag{5·12}$$

where we introduced the notation

$$\big|G_{\pm}\phi_{\pm}\big\rangle \equiv G_{\frac{1}{2}}|\phi_{+}\rangle + G_{-\frac{1}{2}}|\phi_{-}\rangle\,, \qquad \big|G_{\mp}\phi_{\pm}\big\rangle \equiv G_{-\frac{1}{2}}|\phi_{+}\rangle + G_{\frac{1}{2}}|\phi_{-}\rangle\,. \tag{5·13}$$

Since L_0 is invertible, Eq. (5·12) can be solved as

$$\left[1-\left(L_+\frac{1}{L_0}\right)^2\right]2L_0\,|\phi\rangle = |G_\mp\phi_\pm\rangle - L_+\frac{1}{L_0}\,|G_\pm\phi_\pm\rangle\,,$$

$$\left[1-\left(L_+\frac{1}{L_0}\right)^2\right]2L_0\,|\phi^*\rangle = |G_\pm\phi_\pm\rangle - L_+\frac{1}{L_0}\,|G_\mp\phi_\pm\rangle\,. \tag{5·14}$$

The symplectic metric in this sector

$$\big\langle\phi_+\big|\psi_-\big\rangle + \big\langle\phi_-\big|\psi_+\big\rangle = \big\langle\phi\big|\,G_\pm\psi_\pm\big\rangle + \big\langle\phi^*\big|\,G_\mp\psi_\pm\big\rangle, \tag{5·15}$$

with the same notation for $|\psi_\pm\rangle$ as Eq. (5·13), implies that if we take the component $|G_\mp\psi_\pm\rangle$ as a *field*, then $|\phi^*\rangle$ is the conjugate antifield; indeed,

$$\delta_\mathrm{B}\,|\phi_\pm\rangle = G_{\mp\frac{1}{2}}\,|\phi_0\rangle + \cdots \tag{5·16}$$

implies that we can eliminate $|\phi^*\rangle$ in $|\phi_\pm\rangle$ by the freedom of $|\phi_0\rangle$. However, this BRST transformation also shows that another freedom $|\phi\rangle$ in $|\phi_\pm\rangle$ can no longer be eliminated so that $|\phi\rangle$ should be a *field* and $\big|G_\pm\psi_\pm\big\rangle$ must be an antifield. The BRST transformation of $\big|G_\pm\psi_\pm\big\rangle$ is given by

$$\delta_\mathrm{B}\big|G_\pm\psi_\pm\big\rangle = 2L_0\,|\psi_0\rangle + \cdots\,, \tag{5·17}$$

so that it can actually be eliminated.

We are thus led to the following gauge-fixing conditions in the $(|\phi_\pm\rangle\,, |\psi_\pm\rangle)$ sector:

$$\big|G_\pm\psi_\pm\big\rangle = 0, \quad |\phi^*\rangle \propto |G_\pm\phi_\pm\rangle - L_+\frac{1}{L_0}\,|G_\mp\phi_\pm\rangle = 0; \tag{5·18}$$

the remaining *field* degrees of freedom are

$$|\phi\rangle \qquad \text{and} \qquad |G_\mp\psi_\pm\rangle\ . \tag{5·19}$$

We have thus identified gauge-fixing conditions as well as the fields and antifields. We can now read out the BRST transformation (3·34) of the fields after gauge fixing under which the gauge-fixed action is invariant.

So far so good. However, we have not yet succeeded in finding the propagator in a closed form in this gauge. This is because the gauge-fixed action takes a rather complicated form in terms of the component fields. Therefore, in the next section, we try another approach to gauge fixing, directly working with total string fields, which gives gauge-fixing conditions that are very close to the above conditions in this section.

§6. Gauge fixing of NS sector by nonlocal projection

In this section, we consider another gauge-fixing approach that is not as explicit as the that discussed in the previous section but more suitable for computing the propagator. We only investigate the free theory part which is sufficient for the purpose to study what conditions can be taken as the gauge choice.

6.1. *Iterative gauge fixing*

If we concentrate only on the free theory part, the conventional gauge-fixing procedure can be applied. The free classical action of the NS sector is

$$S_0 = \frac{1}{2}\langle \Phi_{(1)}|Y\bar{Y}Q_{\mathrm{B}}|\Phi_{(1)}\rangle = \frac{1}{2}\langle \hat{\Phi}_{(1)}|Y\bar{Y}Q_{\mathrm{B}}|\hat{\Phi}_{(1)}\rangle, \tag{6·1}$$

where $|\Phi_{(1)}\rangle$ is the classical string field restricted onto the ghost number one. Here, in this section, we project out the kernel degrees of freedom of $Y\bar{Y}$ as the hatted field $|\hat{\Phi}_{(1)}\rangle = \mathcal{P}_{Y\bar{Y}}|\Phi_{(1)}\rangle$ using the BRST-invariant projection operator

$$\mathcal{P}_{Y\bar{Y}} = X_{-\frac{1}{2}}X_{\frac{1}{2}}Y\bar{Y}, \qquad [Q_{\mathrm{B}}, \mathcal{P}_{Y\bar{Y}}] = 0, \tag{6·2}$$

where

$$X_{\pm\frac{1}{2}} = [Q_{\mathrm{B}}, \Theta(\beta_{\pm\frac{1}{2}})] = \delta(\beta_{\pm\frac{1}{2}})G_{\pm\frac{1}{2}} - b_{\pm 1}\delta'(\beta_{\pm\frac{1}{2}}). \tag{6·3}$$

To compute the propagator, this projection operator is more convenient than the $\mathcal{P}_0$ introduced in §4, although its component form is very complicated. Because the operator $X_{-\frac{1}{2}}X_{\frac{1}{2}}$ commutes with both L_0 and b_0, i.e.,

$$[L_0, X_{-\frac{1}{2}}X_{\frac{1}{2}}] = 0, \qquad [b_0, X_{-\frac{1}{2}}X_{\frac{1}{2}}] = 0, \tag{6·4}$$

this is suitable for an analog of the conventional Siegel gauge, as we will see shortly.

The action (6·1) is invariant under the gauge transformation

$$\delta|\hat{\Phi}_{(1)}\rangle = Q_{\mathrm{B}}|\hat{\Lambda}_{(0)}\rangle, \tag{6·5}$$

so we must fix it by choosing an appropriate gauge condition. For this purpose, we generally define a decomposition of the hatted NS field $|\hat{A}\rangle = \mathcal{P}_{Y\bar{Y}}\,|A\rangle$ by

$$\begin{aligned} |\hat{A}\rangle &= \mathcal{P}_{\mathrm{NS}}|\hat{A}\rangle + \mathcal{P}_{\mathrm{NS}}^{\perp}|\hat{A}\rangle \\ &\equiv |\hat{A}_{\parallel}\rangle + |\hat{A}_{\perp}\rangle, \end{aligned} \tag{6·6}$$

where $\mathcal{P}_{\rm NS}$ and $\mathcal{P}_{\rm NS}^{\perp}$ are, respectively, the Siegel-gauge projection operators $\mathcal{P}_b$ and $(1-\mathcal{P}_b)$ restricted in the hatted subspace:

$$\mathcal{P}_b \equiv \frac{b_0}{L_0}Q_{\rm B}, \tag{6·7}$$

$$\mathcal{P}_{\rm NS} = \mathcal{P}_{Y\bar{Y}}\mathcal{P}_b\mathcal{P}_{Y\bar{Y}}, \qquad \mathcal{P}_{\rm NS}^{\perp} = \mathcal{P}_{Y\bar{Y}}(1-\mathcal{P}_b)\mathcal{P}_{Y\bar{Y}} = \mathcal{P}_{\rm NS}^{\dagger}. \tag{6·8}$$

These operators satisfy

$$\mathcal{P}_{Y\bar{Y}} = \mathcal{P}_{\rm NS} + \mathcal{P}_{\rm NS}^{\perp}, \tag{6·9}$$

$$(\mathcal{P}_{\rm NS})^2 = \mathcal{P}_{\rm NS}, \qquad (\mathcal{P}_{\rm NS}^{\perp})^2 = \mathcal{P}_{\rm NS}^{\perp}, \tag{6·10}$$

$$\mathcal{P}_{\rm NS}\mathcal{P}_{\rm NS}^{\perp} = \mathcal{P}_{\rm NS}^{\perp}\mathcal{P}_{\rm NS} = 0, \tag{6·11}$$

$$\mathcal{P}_{Y\bar{Y}}^{\dagger}Y\bar{Y}\mathcal{P}_{\rm NS}^{\perp} = \mathcal{P}_{\rm NS}^{\dagger}Y\bar{Y}\mathcal{P}_{Y\bar{Y}}. \tag{6·12}$$

This decomposition (6·6) using the nonlocal projection operator $\mathcal{P}_b$ in (6·7) splits the hatted string field into halves because the ranks of the projection operators $\mathcal{P}_{\rm NS}$ and $\mathcal{P}_{\rm NS}^{\dagger}$ must be the same. Using the last relation (6·12), which follows from the commutativity (6·4), one can show that this has a nondegenerate and cross-diagonal inner product:

$$\langle \hat{A}_1|Y\bar{Y}|\hat{A}_2\rangle = \langle \hat{A}_{1\perp}|Y\bar{Y}|\hat{A}_{2\|}\rangle + \langle \hat{A}_{1\|}|Y\bar{Y}|\hat{A}_{2\perp}\rangle. \tag{6·13}$$

Although these facts are useful to identify the gauge condition and carry out iterative gauge fixing as we will see below, it should be noted that this decomposition cannot exactly be identified as the field and antifield decompositions as was carried out in the previous section in Eq. (4·11). In this regard, the detailed analysis is given in Appendix C in the R string case, which can be explicitly studied using component field expansion in ghost zero modes.

Note that the gauge transformation (6·5) splits in this decomposition into

$$\delta|\hat{\Phi}_{\|(1)}\rangle = 0, \tag{6·14}$$

$$\delta|\hat{\Phi}_{\perp(1)}\rangle = Q_{\rm B}|\hat{\Lambda}_{\|(0)}\rangle, \tag{6·15}$$

since $\mathcal{P}_{\rm NS}Q_{\rm B} = 0$ and $\mathcal{P}_{\rm NS}^{\perp}Q_{\rm B} = Q_{\rm B}\mathcal{P}_{\rm NS}$. Because of Eq. (6·15), we can gauge away the $|\hat{\Phi}_{\perp(1)}\rangle$ part, that is, we can choose the gauge condition

$$|\hat{\Phi}_{\perp(1)}\rangle = \mathcal{P}_{Y\bar{Y}}(1-\mathcal{P}_b)|\hat{\Phi}_{(1)}\rangle = 0, \tag{6·16}$$

which must be equivalent to setting the antifield equal to zero.

The BRST transformation is defined by replacing the gauge parameter $|\hat{\Lambda}_{(0)}\rangle$ by the ghost string field $|\hat{\Phi}_{(0)}\rangle$:

$$\delta_{\rm B}|\hat{\Phi}_{(1)}\rangle = Q_{\rm B}|\hat{\Phi}_{(0)}\rangle. \tag{6·17}$$

To construct the gauge-fixed action, it is convenient to also introduce the antighost field $|\hat{\Phi}_{(2)}\rangle$ and the NL field $|\hat{B}_{(2)}\rangle$ with the BRST transformation

$$\delta_{\mathrm{B}}|\hat{\Phi}_{(2)}\rangle = |\hat{B}_{(2)}\rangle, \qquad \delta_{\mathrm{B}}|\hat{B}_{(2)}\rangle = 0. \tag{6·18}$$

The gauge-fixing and FP ghost action can be given by using these fields as

$$\begin{aligned} S_1 &= -\delta_{\mathrm{B}}\left(\langle\hat{\Phi}_{(2)}|Y\bar{Y}(1-\mathcal{P}_b)|\hat{\Phi}_{(1)}\rangle\right) \\ &= -\langle\hat{B}_{(2)}|Y\bar{Y}(1-\mathcal{P}_b)|\hat{\Phi}_{(1)}\rangle + \langle\hat{\Phi}_{(2)}|\mathcal{P}_b^{\dagger}Y\bar{Y}Q_{\mathrm{B}}|\hat{\Phi}_{(0)}\rangle. \end{aligned} \tag{6·19}$$

Here, we have used the relation (6·12). We can eliminate the NL field and rewrite the total action in the first step in the form

$$S_0 + S_1 = \frac{1}{2}\langle\hat{\Phi}_{\parallel(1)}|Y\bar{Y}Q_{\mathrm{B}}|\hat{\Phi}_{\parallel(1)}\rangle + \langle\hat{\Phi}_{\parallel(2)}|Y\bar{Y}Q_{\mathrm{B}}|\hat{\Phi}_{(0)}\rangle. \tag{6·20}$$

Here, it must be again noted that $|\hat{\Phi}_{\parallel(1)}\rangle$ is not purely the field component but also contains the antifield component, although we do not need to consider it to compute the propagator. However, we must set its antifield component equal to zero to obtain the BRST transformation of the (gauge-fixed) field because $\delta_{\mathrm{B}}|\hat{\Phi}_{\parallel(1)}\rangle = 0$. (See Appendix C for detailed analysis in the R string case.)

The first step action (6·20) still has a new gauge invariance under the transformation

$$\delta|\hat{\Phi}_{(0)}\rangle = Q_{\mathrm{B}}|\hat{\Lambda}_{(-1)}\rangle, \tag{6·21}$$

which can be similarly fixed by introducing a set of (ghost for) ghost, NL, and antighost fields, $(|\hat{\Phi}_{(-1)}\rangle, |\hat{B}_{(3)}\rangle, |\hat{\Phi}_{(3)}\rangle)$. We can repeat the same procedure as the first step. We must infinitely repeat the procedure; however, we can easily see that the final form of the gauge-fixed action is

$$S_{GF} = \frac{1}{2}\langle\hat{\Phi}_{\parallel}|Y\bar{Y}Q_{\mathrm{B}}|\hat{\Phi}_{\parallel}\rangle, \tag{6·22}$$

where $|\hat{\Phi}_{\parallel}\rangle$ defined by

$$|\hat{\Phi}_{\parallel}\rangle = \sum_{g=-\infty}^{\infty}|\hat{\Phi}_{\parallel(g)}\rangle, \tag{6·23}$$

is the gauge-fixed string field with the ghost number constraint relaxed.*)

*) Strictly speaking, we must set its antifield component equal to zero, as already mentioned.

6.2. *Propagator*

We can also derive the propagator for the gauge condition (6·16). Let us start by adding a source term to the free gauge-fixed action (6·22):

$$\begin{aligned} S[\Phi, J] &= \frac{1}{2}\langle\hat{\Phi}_{\|}|Y\bar{Y}Q_{\mathrm{B}}|\hat{\Phi}_{\|}\rangle - \langle\hat{\Phi}_{\|}|J\rangle, \\ &= \frac{1}{2}\langle\Phi|\mathcal{P}_{\mathrm{NS}}^{\dagger}Y\bar{Y}Q_{\mathrm{B}}\mathcal{P}_{\mathrm{NS}}|\Phi\rangle - \langle\Phi|\mathcal{P}_{\mathrm{NS}}^{\dagger}|J\rangle. \end{aligned} \tag{6·24}$$

We can complete the square of this expression by solving the equation of motion

$$\mathcal{P}_{\mathrm{NS}}^{\dagger}Y\bar{Y}Q_{\mathrm{B}}|\hat{\Phi}_{\|}\rangle = \mathcal{P}_{\mathrm{NS}}^{\dagger}|J\rangle, \tag{6·25}$$

as

$$|\hat{\Phi}_{\|}\rangle = \mathcal{P}_{\mathrm{NS}}\frac{b_0X_{-\frac{1}{2}}X_{\frac{1}{2}}}{L_0}\mathcal{P}_{Y\bar{Y}}^{\dagger}|J\rangle. \tag{6·26}$$

The action (6·24) is then rewritten as

$$S[\Phi, J] = \frac{1}{2}\langle\hat{\Phi}_{\|}'|Y\bar{Y}Q_{\mathrm{B}}|\hat{\Phi}_{\|}'\rangle - \frac{1}{2}\langle J|\mathcal{P}_{\mathrm{NS}}\frac{b_0X_{-\frac{1}{2}}X_{\frac{1}{2}}}{L_0}\mathcal{P}_{Y\bar{Y}}^{\dagger}|J\rangle, \tag{6·27}$$

with $|\hat{\Phi}_{\|}'\rangle = |\hat{\Phi}_{\|}\rangle - \Pi_{\mathrm{NS}}|J\rangle$. The propagator Π_{NS} is thus found to be

$$\Pi_{\mathrm{NS}} = \mathcal{P}_{\mathrm{NS}}\frac{b_0X_{-\frac{1}{2}}X_{\frac{1}{2}}}{L_0}\mathcal{P}_{Y\bar{Y}}^{\dagger} = \mathcal{P}_{Y\bar{Y}}\frac{b_0X_{-\frac{1}{2}}X_{\frac{1}{2}}}{L_0}\mathcal{P}_{\mathrm{NS}}^{\dagger}. \tag{6·28}$$

This has quite a reasonable form as a counterpart of the Ramond-Siegel gauge propagator (3·50) in the R sector.

§7. Gauge fixing of total interacting theory

Now, it is straightforward to fix the gauge symmetry of the total interacting theory including both the R and NS sectors with the Yukawa coupling term $\Phi\Psi\Psi$. The total action S satisfying the BV master equation is determined by the variational equation

$$\delta S = \int Y\bar{Y}\delta\Phi * \delta_{\mathrm{B}}\Phi + \int Y\delta\Psi * \delta_{\mathrm{B}}\Psi, \tag{7·1}$$

with the nonlinear BRST transformation

$$\delta_{\mathrm{B}}\Phi = Q_{\mathrm{B}}\Phi + \Phi * \Phi + \bar{X}\Psi * \Psi, \tag{7·2}$$
$$\delta_{\mathrm{B}}\Psi = Q_{\mathrm{B}}\Psi + \Phi * \Psi + \Psi * \Phi. \tag{7·3}$$

The solution is easily found to be

$$S = \frac{1}{2}\int Y\bar{Y}\,\Phi * Q_{\rm B}\Phi + \frac{1}{3}\int Y\bar{Y}\,\Phi * \Phi * \Phi + \frac{1}{2}\int Y\,\Psi * Q_{\rm B}\Psi + \int Y\Phi * \Psi * \Psi. \tag{7·4}$$

This has the same form as the gauge-invariant action (2·1), but now the ghost number constraints on the string fields Φ and Ψ are relaxed. As in the case with the free theory, we can restrict the string fields Φ and Ψ to the hatted ones $\hat{\Phi} = \mathcal{P}_{Y\bar{Y}}\Phi$ and $\hat{\Psi} = \mathcal{P}_Y\Psi$, because of the presence of the inverse picture-changing operators Y and $Y\bar{Y}$ at the interaction terms and also the property of the NS-R-R vertex

$$\langle V(1_{\rm NS}, 2_{\rm R}, 3_{\rm R})|\mathcal{P}^{(1)}_{Y\bar{Y}} = \langle V(1_{\rm NS}, 2_{\rm R}, 3_{\rm R})|, \tag{7·5}$$

resulting from Eq. (2·9). The gauge-fixed action is finally obtained as

$$\begin{aligned} S = &\frac{1}{2}\int Y\bar{Y}\,\hat{\Phi}_{\|} * Q_{\rm B}\hat{\Phi}_{\|} + \frac{1}{3}\int Y\bar{Y}\,\hat{\Phi}_{\|} * \hat{\Phi}_{\|} * \hat{\Phi}_{\|} \\ &+ \frac{1}{2}\int Y\,\hat{\Psi}_{\|} * Q_{\rm B}\hat{\Psi}_{\|} + \int Y\hat{\Phi}_{\|} * \hat{\Psi}_{\|} * \hat{\Psi}_{\|}\ , \end{aligned} \tag{7·6}$$

where $\hat{\Phi}_{\|}$ and $\hat{\Psi}_{\|}$ are the gauge-fixed fields obtained by setting the antifields equal to zero corresponding to the gauge conditions, that is, Eqs. (5·10) and (5·18) or (6·16) for the NS field $\hat{\Phi}_{\|}$ and Eq. (3·29) for the R field $\hat{\Psi}_{\|}$.

§8. Discussion

For the NS sector, we have considered two different gauge-fixing approaches, each of which has both merits and demerits. In §5, we studied the problem using component fields. We explicitly chose a set of antifields set equal to zero as gauge conditions:

$$|\psi_{+-}\rangle = |\phi_{+-}\rangle = 0, \tag{8·1}$$

$$|G_\pm\psi_\pm\rangle = 0, \tag{8·2}$$

$$|G_\pm\phi_\pm\rangle - L_+\frac{1}{L_0}|G_\mp\phi_\pm\rangle = 0. \tag{8·3}$$

Because the latter two conditions can be written as

$$|\psi^*\rangle = -\frac{1}{L_0}L_+|\psi\rangle, \tag{8·4}$$

$$|\phi^*\rangle = 0, \tag{8·5}$$

respectively, using the decomposition (5·11) and a similar one for $|\psi_\pm\rangle$, we can, in principle, compute the BRST transformation of the *fields* $|\phi_0\rangle$, $|\psi_0\rangle$, $|\phi\rangle$, and $|G_\mp\psi_\pm\rangle \propto |\psi\rangle$ from the BRST transformation (A·14). We could not, however, find the propagator because the explicit form of the kinetic term is complicated in the component form.

We presented another gauge-fixing approach using the nonlocal projection operator. We found the gauge condition

$$|\hat{\Phi}_\perp\rangle = \mathcal{P}_{Y\bar{Y}}\left(1 - \frac{b_0}{L_0}Q_\mathrm{B}\right)\mathcal{P}_{Y\bar{Y}}|\hat{\Phi}\rangle = 0, \tag{8·6}$$

which is suitable for computing the propagator. The propagator in this gauge is given by

$$\Pi_\mathrm{NS} = \mathcal{P}_\mathrm{NS}\frac{b_0 X_{-\frac{1}{2}}X_{\frac{1}{2}}}{L_0}\mathcal{P}^\dagger_{Y\bar{Y}}, \tag{8·7}$$

in quite a parallel form with the R field propagator

$$\Pi_\mathrm{R} = \frac{b_0 X_0}{L_0}. \tag{8·8}$$

It should also be noted that the propagator (8·7) can be rewritten in the form

$$\Pi_\mathrm{NS} = \mathcal{P}_{Y\bar{Y}}\frac{b_0}{L_0}WQ_\mathrm{B}\frac{b_0}{L_0}\mathcal{P}^\dagger_{Y\bar{Y}} \tag{8·9}$$

with

$$W = \left(X_{-\frac{1}{2}}X_{\frac{1}{2}}Y\bar{Y}X_{-\frac{1}{2}}X_{\frac{1}{2}}\right), \tag{8·10}$$

which is the counterpart of that discussed in Ref. 5), although the kernel problem was not considered. Because

$$\begin{aligned}\Pi_\mathrm{NS} &= \mathcal{P}_{Y\bar{Y}}\left(\frac{b_0 X_{-\frac{1}{2}}X_{\frac{1}{2}}}{L_0} - \frac{b_0}{L_0}W\frac{b_0}{L_0}Q_\mathrm{B}\right)\mathcal{P}^\dagger_{Y\bar{Y}} \\ &= \mathcal{P}_{Y\bar{Y}}\left(\frac{b_0 X_{-\frac{1}{2}}X_{\frac{1}{2}}}{L_0} - Q_\mathrm{B}\frac{b_0}{L_0}W\frac{b_0}{L_0}\right)\mathcal{P}^\dagger_{Y\bar{Y}},\end{aligned} \tag{8·11}$$

one can see that the physical amplitudes have only the desired pole at $L_0 = 0$. This is very favorable in computing perturbative amplitudes.

On the other hand, we cannot determine the BRST transformation of the gauge-fixed field in this approach because $|\hat{\Phi}_\parallel\rangle$ is BRST-invariant:

$$\delta_\mathrm{B}|\hat{\Phi}_\parallel\rangle = 0. \tag{8·12}$$

This is because $|\hat{\Phi}_\parallel\rangle$ is not the field itself but also contains the antifield components, as explained in Appendix C for the R sector. For the NS sector, however, we have not yet identified the field and antifield components in $|\hat{\Phi}_\parallel\rangle$, because its component-field form is very complicated.

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Appendix A
—— *BRST Operators and BRST Transformations* ——

A.1. *Ramond string*

We expand the BRST operator for the R string in the ghost zero mode operators c_0, b_0, γ_0, and β_0:

$$Q_{\rm B} = c_0 L_0 + b_0 M + \gamma_0 F + \beta_0 K + \tilde{Q}_{\rm B} - \gamma_0^2 b_0, \tag{A·1}$$

where

$$L_0 = L_0^{(\rm m)} + \sum_{n\neq 0} n : c_{-n} b_n : + \sum_{n\neq 0} n : \beta_{-n}\gamma_n :, \tag{A·2a}$$

$$M = -\sum_{n\neq 0} n c_{-n} c_n - \sum_{n\neq 0} \gamma_{-n}\gamma_n, \tag{A·2b}$$

$$F = F_0^{(\rm m)} - \sum_{n\neq 0} \frac{1}{2} n \beta_{-n} c_n - 2\sum_{n\neq 0} b_{-n}\gamma_n, \tag{A·2c}$$

$$K = \sum_{n\neq 0} \frac{3}{2} c_{-n}\gamma_n, \tag{A·2d}$$

$$\begin{aligned}\tilde{Q}_{\rm B} = & \sum_{n\neq 0} c_{-n} L_n^{(\rm m)} + \sum_{n\neq 0} \gamma_{-n} F_n^{(\rm m)} + \sum_{\substack{m,n \\ m+n\neq 0}} \frac{1}{2}(n-m) b_{-n-m} c_n c_m \\ & + \sum_{\substack{m,n \\ m+n\neq 0}} \frac{1}{2}(2m-n)\beta_{-n-m} c_n \gamma_m - \sum_{\substack{m,n \\ m+n\neq 0}} \gamma_{-n}\gamma_{-m} b_{n+m},\end{aligned} \tag{A·2e}$$

with $L_n^{(\rm m)}$ and $F_n^{(\rm m)}$ being the super-Virasoro operators for the matter part. Note that the hermitian conjugate of the BRST operator for the R string is given as[20]

$$Q_{\rm B}^\dagger = \psi_0^0 Q_{\rm B} \psi_0^0. \tag{A·3}$$

A.2. *Neveu-Schwarz string*

Let us call the ghost and superghost modes $c_0, c_\pm, b_0, b_\pm, \gamma_{\pm\frac{1}{2}}$, and $\beta_{\pm\frac{1}{2}}$ 'zero modes' collectively in this NS sector. We expand the BRST operator for the NS string in these 'zero modes':

$$\begin{aligned}
Q_\mathrm{B} = c_0 &\left(\tilde{L}_0 - c_+b_- - c_-b_+ + \frac{1}{2}\gamma_{\frac{1}{2}}\beta_{-\frac{1}{2}} - \frac{1}{2}\gamma_{-\frac{1}{2}}\beta_{\frac{1}{2}}\right) + b_0\left(\tilde{M} + c_-c_+ - 2\gamma_{-\frac{1}{2}}\gamma_{\frac{1}{2}}\right) \\
&+ c_+\tilde{L}_+ + c_-\tilde{L}_- + \tilde{M}_+b_+ + \tilde{M}_-b_- + \gamma_{-\frac{1}{2}}\tilde{G}_{\frac{1}{2}} + \gamma_{\frac{1}{2}}\tilde{G}_{-\frac{1}{2}} + \tilde{K}_{\frac{1}{2}}\beta_{-\frac{1}{2}} + \tilde{K}_{-\frac{1}{2}}\beta_{\frac{1}{2}} \\
&+ \frac{3}{2}c_{-2}\left(-c_+b_+ - c_-b_+ + c_+b_- + c_-b_-\right) + \frac{3}{2}c_2\left(c_+b_+ - c_-b_+ + c_+b_- - c_-b_-\right) \\
&+ \frac{1}{2}c_+\left(\gamma_{\frac{1}{2}}\beta_{\frac{1}{2}} - \gamma_{-\frac{1}{2}}\beta_{-\frac{1}{2}}\right) - \frac{1}{2}c_-\left(\gamma_{\frac{1}{2}}\beta_{\frac{1}{2}} + \gamma_{-\frac{1}{2}}\beta_{-\frac{1}{2}}\right) + c_+\left(\gamma_{\frac{3}{2}}\beta_{-\frac{1}{2}} - \gamma_{-\frac{3}{2}}\beta_{\frac{1}{2}}\right) \\
&- c_-\left(\gamma_{\frac{3}{2}}\beta_{-\frac{1}{2}} + \gamma_{-\frac{3}{2}}\beta_{\frac{1}{2}}\right) - \left(\gamma_{\frac{1}{2}}^2 + \gamma_{-\frac{1}{2}}^2\right)b_+ + \left(\gamma_{-\frac{1}{2}}^2 - \gamma_{\frac{1}{2}}^2\right)b_- \\
&- 2\left(\gamma_{\frac{1}{2}}\gamma_{-\frac{3}{2}} + \gamma_{-\frac{1}{2}}\gamma_{\frac{3}{2}}\right)b_+ + 2\left(\gamma_{\frac{1}{2}}\gamma_{-\frac{3}{2}} - \gamma_{-\frac{1}{2}}\gamma_{\frac{3}{2}}\right)b_- + \tilde{Q}_\mathrm{B}.
\end{aligned} \tag{A·4}$$

Here, the tilde symbols on L_n, M_n, G_r, K_r, and Q_B denote the operators with all the parts containing 'zero mode' operators omitted:

$$\tilde{L}_0 = L_0^{(\mathrm{m})} + \sum_{n\neq 0,\pm 1} n :b_{-n}c_n: + \sum_{r\neq\pm\frac{1}{2}} r :\beta_{-r}\gamma_r:, \tag{A·5a}$$

$$\tilde{L}_1 = L_1^{(\mathrm{m})} + \sum_{n\neq 0,\pm 1,2} (n+1)b_{-n+1}c_n + \sum_{r\neq\pm\frac{1}{2},\frac{3}{2}} \frac{1}{2}(2r+1)\beta_{-r+1}\gamma_r, \tag{A·5b}$$

$$\tilde{L}_{-1} = L_{-1}^{(\mathrm{m})} + \sum_{n\neq 0,\pm 1,-2} (n-1)b_{-n-1}c_n + \sum_{r\neq\pm\frac{1}{2},-\frac{3}{2}} \frac{1}{2}(2r-1)\beta_{-r-1}\gamma_r, \tag{A·5c}$$

$$\tilde{L}_\pm = \frac{1}{2}\left(\tilde{L}_{-1} \pm \tilde{L}_1\right), \tag{A·5d}$$

$$\tilde{M} = -\sum_{n\neq 0,\pm 1} nc_{-n}c_n - \sum_{r\neq\pm\frac{1}{2}} \gamma_{-r}\gamma_r, \tag{A·5e}$$

$$\tilde{M}_1 = -\sum_{n\neq 0,\pm 1,2} \frac{1}{2}(2n-1)c_{-n+1}c_n - \sum_{r\neq\pm\frac{1}{2},\frac{3}{2}} \gamma_{-r+1}\gamma_r, \tag{A·5f}$$

$$\tilde{M}_{-1} = -\sum_{n\neq 0,\pm 1,-2} \frac{1}{2}(2n+1)c_{-n-1}c_n - \sum_{r\neq\pm\frac{1}{2},-\frac{3}{2}} \gamma_{-r-1}\gamma_r, \tag{A·5g}$$

$$\tilde{M}_\pm = \tilde{M}_1 \pm \tilde{M}_{-1}, \tag{A·5h}$$

$$\tilde{G}_{\pm\frac{1}{2}} = G_{\pm\frac{1}{2}}^{(\mathrm{m})} - \sum_{n\neq 0,\pm 1} \left(\frac{1}{2}(n\pm 1)\beta_{-n\pm\frac{1}{2}}c_n + b_{-n}\gamma_{n\pm\frac{1}{2}}\right), \tag{A·5i}$$

$$\tilde{K}_{\pm\frac{1}{2}} = \sum_{n\neq 0,\pm 1} \frac{1}{2}(-3n\pm 1)c_n\gamma_{-n\pm\frac{1}{2}}, \tag{A·5j}$$

$$\tilde{Q}_{\mathrm{B}} = \sum_{n\neq 0,\pm 1} c_{-n} L_n^{(\mathrm{m})} + \sum_{r\neq \pm\frac{1}{2}} \gamma_{-r} G_r^{(\mathrm{m})} + \sum_{\substack{m,n \\ m+n\neq 0,\pm 1}} \frac{1}{2}(n-m) b_{-n-m} c_n c_m$$
$$+ \sum_{n\neq 0,\pm 1} \sum_{r\neq\pm\frac{1}{2}} \left(\frac{1}{2}(2r-n)\beta_{-n-r} c_n \gamma_r - b_{-n}\gamma_{n-r}\gamma_r \right). \qquad \text{(A·5k)}$$

The translated BRST operator by T in Eq. (4·3) can be obtained as

$$\begin{aligned} e^T Q_{\mathrm{B}}\, e^{-T} = \tilde{\mathcal{Q}} &+ \frac{1}{2}\gamma_{\frac{1}{2}}(\partial C_+ - C_-)\beta_{-\frac{1}{2}} + \frac{1}{2}\gamma_{-\frac{1}{2}}(\partial C_+ - C_-)\beta_{\frac{1}{2}} \\ &+ \frac{1}{2}\gamma_{\frac{1}{2}}(\partial C_- + C_+)\beta_{\frac{1}{2}} - \frac{1}{2}\gamma_{-\frac{1}{2}}(\partial C_- + C_+)\beta_{-\frac{1}{2}} + \gamma_{\frac{1}{2}}\tilde{G}_{-\frac{1}{2}} + \gamma_{-\frac{1}{2}}\tilde{G}_{\frac{1}{2}} \\ &+ \tilde{\mathcal{M}} b_+ + 2\gamma_{\frac{1}{2}}\left(\mathit{\Gamma}_+ - \gamma_{-\frac{3}{2}}\right) b_+ + 2\gamma_{-\frac{1}{2}}\left(\mathit{\Gamma}_- - \gamma_{\frac{3}{2}}\right) b_+ - \left(\gamma_{\frac{1}{2}}^2 + \gamma_{-\frac{1}{2}}^2\right) b_+ \\ &+ \frac{3}{2} c_+(c_{-2} - c_2) b_+ + c_+ \tilde{L}_+ + \frac{1}{2} c_+ \gamma_{\frac{1}{2}}\beta_{\frac{1}{2}} - \frac{1}{2} c_+\gamma_{-\frac{1}{2}}\beta_{-\frac{1}{2}}, \end{aligned} \qquad \text{(A·6)}$$

with

$$\tilde{\mathcal{Q}} = \tilde{Q}_{\mathrm{B}} - C_- \tilde{L}_0 - C_+ \tilde{L}_- - \mathit{\Gamma}_- \tilde{G}_{\frac{1}{2}} - \mathit{\Gamma}_+ \tilde{G}_{-\frac{1}{2}}, \qquad \text{(A·7)}$$
$$\tilde{\mathcal{M}} = \tilde{M}_+ - C_- C_+ + \frac{3}{2}(c_{-2} + c_2) C_+ - \mathit{\Gamma}_+^2 - \mathit{\Gamma}_-^2 + 2\mathit{\Gamma}_-\gamma_{\frac{3}{2}} + 2\mathit{\Gamma}_+\gamma_{-\frac{3}{2}}, \qquad \text{(A·8)}$$
$$\partial c_+(i) = \frac{1}{2}\big(\partial c(i) + \partial c(-i)\big) = c_0 + \partial C_+, \qquad \text{(A·9)}$$
$$\partial C_+ = \sum_{k=1}^{\infty} (-1)^k \big((-2k+1) c_{2k} + (2k+1) c_{-2k}\big), \qquad \text{(A·10)}$$
$$\partial c_-(i) = \frac{1}{2i}\big(\partial c(i) - \partial c(-i)\big) = 2c_{-1} + \partial C_-, \qquad \text{(A·11)}$$
$$\partial C_- = i\sum_{k=1}^{\infty} (-1)^k \big(2k c_{2k+1} + 2(k+1) c_{-2k-1}\big). \qquad \text{(A·12)}$$

Note that in obtaining Eq. (A·6) we have eliminated the terms containing c_0 and c_- on the left or b_0 and b_- on the right. This is because we use it only when the $(Y\bar{Y})_0$ operator is present on the left and the reduced field $e^T \mathcal{P}_0 \mathit{\Phi}$ on the right.

The expression of the BRST transformation in component fields can be most easily obtained by computing

$$e^T\left(\delta_{\mathrm{B}}\tilde{\mathit{\Phi}}\right) = e^T \mathcal{P}_0\, e^{-T}\cdot e^T Q_{\mathrm{B}}\, e^{-T}\cdot e^T \mathcal{P}_0 \mathit{\Phi}, \qquad \text{(A·13)}$$

using Eq. (A·6). The result is

$$\delta_{\mathrm{B}}\phi_0 = \tilde{\mathcal{Q}}\phi_0 + \tilde{\mathcal{M}}\psi_0, \qquad \text{(A·14a)}$$
$$\delta_{\mathrm{B}}\phi_\pm = \tilde{G}_{\mp\frac{1}{2}}\phi_0 + \left(\tilde{\mathcal{Q}} - \frac{1}{2}(\partial C_+ - C_-)\right)\phi_\pm \mp \frac{1}{2}(\partial C_- + C_+)\phi_\mp$$

$$+\,2\left(\Gamma_{+}-\gamma_{-\frac{3}{2}}\right)\psi_{0}+\tilde{\mathcal{M}}\psi_{\pm}, \tag{A·14b}$$

$$\begin{aligned}\delta_{\mathrm{B}}\phi_{+-}&=\tilde{G}_{-\frac{1}{2}}\phi_{+}+\tilde{G}_{\frac{1}{2}}\phi_{-}+\left(\tilde{\mathcal{Q}}-\partial C_{+}+C_{-}\right)\phi_{+-}\\&\quad-2\psi_{0}+2\left(\Gamma_{+}-\gamma_{-\frac{3}{2}}\right)\psi_{+}+2\left(\Gamma_{-}-\gamma_{\frac{3}{2}}\right)\psi_{-}+\tilde{\mathcal{M}}\psi_{+-},\end{aligned} \tag{A·14c}$$

$$\delta_{\mathrm{B}}\psi_{0}=-\tilde{L}_{+}\phi_{0}+\left(\tilde{\mathcal{Q}}+\frac{3}{2}(c_{2}-c_{-2})\right)\psi_{0}, \tag{A·14d}$$

$$\begin{aligned}\delta_{\mathrm{B}}\psi_{\pm}&=-\tilde{L}_{+}\phi_{\pm}\pm\frac{1}{2}\phi_{\mp}+\tilde{G}_{\mp\frac{1}{2}}\psi_{0}\mp\frac{1}{2}(\partial C_{-}+C_{+})\psi_{\mp}\\&\quad+\left(\tilde{\mathcal{Q}}-\frac{1}{2}(\partial C_{+}-C_{-})+\frac{3}{2}(c_{2}-c_{-2})\right)\psi_{\pm},\end{aligned} \tag{A·14e}$$

$$\begin{aligned}\delta_{\mathrm{B}}\psi_{+-}&=-\tilde{L}_{+}\phi_{+-}+\tilde{G}_{-\frac{1}{2}}\psi_{+}+\tilde{G}_{\frac{1}{2}}\psi_{-}\\&\quad+\left(\tilde{\mathcal{Q}}-\partial C_{+}+C_{-}+\frac{3}{2}(c_{2}-c_{-2})\right)\psi_{+-}.\end{aligned} \tag{A·14f}$$

Appendix B
—— *Picture Changing Operators* ——

The picture-changing operator $X(z)$ and the inverse picture-changing operator $Y(z)$ are defined as

$$X(z)=[Q_{\mathrm{B}},\Theta(\beta(z))]=G(z)\delta(\beta(z))-\partial b\delta'(\beta(z)), \tag{B·1}$$

$$Y(z)=c(z)\delta'(\gamma(z)), \tag{B·2}$$

where $G(z)$ denotes the supercurrent

$$G(z)=G^{(\mathrm{m})}(z)+c\partial\beta(z)+\frac{3}{2}(\partial c)\beta(z)-2\gamma b(z). \tag{B·3}$$

The operators $X(z)$ and $Y(z)$ are the inverses of each other and change the picture numbers by $+1$ and -1, respectively. We write in particular the (inverse) picture-changing operators inserted at the midpoint or its mirror point $z=\pm i$ as

$$Y=Y(i),\quad \bar{Y}=Y(-i),\quad X=X(i),\quad \bar{X}=X(-i). \tag{B·4}$$

The ‘mode versions’ of the (inverse) picture-changing operators are defined by

$$X_{0}=[Q_{\mathrm{B}},\Theta(\beta_{0})]=\delta(\beta_{0})F_{0}-b_{0}\delta'(\beta_{0}), \tag{B·5}$$

$$X_{\pm\frac{1}{2}}=[Q_{\mathrm{B}},\Theta(\beta_{\pm\frac{1}{2}})]=\delta(\beta_{\pm\frac{1}{2}})G_{\pm\frac{1}{2}}-b_{\pm1}\delta'(\beta_{\pm\frac{1}{2}}), \tag{B·6}$$

and

$$Y_{0}=c_{0}\delta'(\gamma_{0}), \tag{B·7}$$

$$(Y\bar{Y})_{0}=\frac{1}{4}c_{-}c_{0}\left(\delta''(\gamma_{\frac{1}{2}})\delta(\gamma_{-\frac{1}{2}})+\delta(\gamma_{\frac{1}{2}})\delta''(\gamma_{-\frac{1}{2}})\right). \tag{B·8}$$

They satisfy the relations:

$$Y(z)X_0Y(z) = Y(z), \qquad X_0Y(z)X_0 = X_0, \tag{B·9a}$$
$$Y_0X_0Y_0 = Y_0, \qquad X_0Y_0X_0 = X_0, \tag{B·9b}$$

and

$$Y\bar{Y}(z)X_{-\frac{1}{2}}X_{\frac{1}{2}}Y\bar{Y}(z) = Y\bar{Y}(z). \tag{B·10}$$

In addition, X_0 and $X_{-\frac{1}{2}}X_{\frac{1}{2}}$ (anti-)commute with b_0 and L_0:

$$\{X_0, b_0\} = 0, \qquad [X_0, L_0] = 0, \tag{B·11}$$
$$\{X_{\pm\frac{1}{2}}, b_0\} = 0, \qquad [X_{\pm\frac{1}{2}}, L_0] = \pm\frac{1}{2}X_{\pm\frac{1}{2}}. \tag{B·12}$$

The hermitian conjugates of the (inverse) picture-changing operators are given as

$$Y(\pm i)^\dagger = Y(\pm i), \tag{B·13}$$
$$X_0^\dagger = \psi_0^0 X_0 \psi_0^0, \tag{B·14}$$
$$(X_{-\frac{1}{2}}X_{\frac{1}{2}})^\dagger = X_{-\frac{1}{2}}X_{\frac{1}{2}}, \tag{B·15}$$

where $\psi_0^{\mu=0}$ is the zero mode of the time component of the matter operator $\psi^\mu(z)$ in the R sector. Accordingly, we have to distinguish the hermitian conjugate and the transposition of the Ramond projection $\mathcal{P}_Y$ given as

$$\mathcal{P}_Y^\dagger = YX_0^\dagger = Y\psi_0^0 X_0 \psi_0^0, \qquad \mathcal{P}_Y^T = YX_0, \tag{B·16}$$

and can confirm the relations

$$\psi_0^0 \mathcal{P}_Y^\dagger = \mathcal{P}_Y^T \psi_0^0, \qquad \mathcal{P}_Y^\dagger \psi_0^0 = \psi_0^0 \mathcal{P}_Y^T. \tag{B·17}$$

Appendix C
—— *Ramond Analog of Decomposition* (6·6) ——

In this appendix, we explain how the decomposition using the nonlocal projection operator gives the proper gauge condition and the propagator for the R string, which can be explicitly studied by component analysis. The nonlocal projection operator of the R string has the form

$$\mathcal{P}_b = \frac{b_0}{L_0} Q_{\rm B}, \tag{C·1}$$
$$= b_0 c_0 - \frac{1}{L_0}(F\gamma_0 + K\beta_0 + \tilde{Q}_{\rm B})b_0. \tag{C·2}$$

Unlike the NS string, this satisfies

$$\mathcal{P}_Y\mathcal{P}_b\mathcal{P}_Y = \mathcal{P}_b\mathcal{P}_Y. \tag{C·3}$$

The decomposition of the projected field $|\hat{\Psi}\rangle = \mathcal{P}_Y|\Psi\rangle$ is defined by

$$|\hat{\Psi}\rangle = \mathcal{P}_b|\hat{\Psi}\rangle + (1-\mathcal{P}_b)|\hat{\Psi}\rangle, \tag{C·4}$$

$$\equiv |\hat{\Psi}_\parallel\rangle + |\hat{\Psi}_\perp\rangle, \tag{C·5}$$

with which the gauge transformation $\delta|\hat{\Psi}\rangle = Q_\mathrm{B}|\hat{\Lambda}\rangle$ splits into

$$\delta|\hat{\Psi}_\parallel\rangle = 0, \tag{C·6}$$

$$\delta|\hat{\Psi}_\perp\rangle = Q_\mathrm{B}|\Lambda_\parallel\rangle. \tag{C·7}$$

Therefore, we can gauge away the $|\hat{\Psi}_\perp\rangle$ part and take the gauge condition

$$|\hat{\Psi}_\perp\rangle = 0. \tag{C·8}$$

As explained in §3, the constrained string field $|\hat{\Psi}\rangle$ can be expressed by the component field as

$$|\hat{\Psi}\rangle = |\phi\rangle - (\gamma_0 + c_0F)|\psi\rangle. \tag{C·9}$$

The decomposed fields $|\hat{\Psi}_\parallel\rangle$ and $|\hat{\Psi}_\perp\rangle$ are also represented by $|\phi\rangle$ and $|\psi\rangle$ as

$$|\hat{\Psi}_\parallel\rangle = |\phi\rangle + \tilde{Q}_\mathrm{B}\frac{F}{L_0}|\psi\rangle, \tag{C·10}$$

$$|\hat{\Psi}_\perp\rangle = -\left(\gamma_0 + c_0F + \tilde{Q}_\mathrm{B}\frac{F}{L_0}\right)|\psi\rangle. \tag{C·11}$$

Since this (C·11) also leads to $|\psi\rangle = \beta_0\big|\hat{\Psi}_\perp\big\rangle$, the gauge condition (C·8) is equivalent to the gauge condition $|\psi\rangle = 0$ used in §3.

We can also obtain the same propagator as that in §3 using this argument; indeed,

$$S = -\frac{1}{2}\langle\hat{\Psi}_\parallel|Y_0Q_\mathrm{B}|\hat{\Psi}_\parallel\rangle + \langle\hat{\Psi}_\parallel|J\rangle, \tag{C·12}$$

$$= \frac{1}{2}\left(\langle J|\frac{X_0b_0}{L_0} - \langle\hat{\Psi}_\parallel|\right)Y_0Q_\mathrm{B}\left(|\hat{\Psi}_\parallel\rangle - \frac{b_0X_0}{L_0}|J\rangle\right) - \frac{1}{2}\langle J|\frac{b_0X_0}{L_0}|J\rangle. \tag{C·13}$$

We can see that the component $|\hat{\Psi}_\parallel\rangle$ (C·10) contains not only the field $|\phi\rangle$ but also the antifield $|\psi\rangle$.The proper BRST transformation of the gauge-fixed field can be obtained only when the antifield component is set equal to zero before the transformation:

$$\delta_\mathrm{B}\left(|\hat{\Psi}_\parallel\rangle|_{|\psi\rangle=0}\right) = \tilde{Q}_\mathrm{B}\left(|\hat{\Psi}_\parallel\rangle|_{|\psi\rangle=0}\right). \tag{C·14}$$

Otherwise, the BRST transformation of $\big|\hat{\Psi}_\parallel\big\rangle$ is zero.

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